IRRESISTIBLE INTEGRALS
Symbolics, Analysis and Experiments in the Evaluation of Integrals

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Factorials and Binomial Coefficients

1.1. Introduction

In this chapter we discuss several properties of factorials and binomial coefficients. These functions will often appear as results of evaluations of definite integrals.

Definition 1.1.1. A function $f : \mathbb{N} \to \mathbb{N}$ is said to satisfy a recurrence if the value $f(n)$ is determined by the values $\{f(1), f(2), \ldots, f(n-1)\}$. The recurrence is of order $k$ if $f(n)$ is determined by the values $\{f(n-1), f(n-2), \ldots, f(n-k)\}$, where $k$ is a fixed positive integer. The notation $f_n$ is sometimes used for $f(n)$.

For example, the Fibonacci numbers $F_n$ satisfy the second-order recurrence

$$F_n = F_{n-1} + F_{n-2}. \quad (1.1.1)$$

Therefore, in order to compute $F_n$, one needs to know only $F_1$ and $F_2$. In this case $F_1 = 1$ and $F_2 = 1$. These values are called the initial conditions of the recurrence. The Mathematica command

$$F[n_] := \text{If}[n==0,1, \text{If}[n==1,1, F[n-1]+F[n-2]]]$$

gives the value of $F_n$. The modified command

$$F[n_] := F[n]= \text{If}[n==0,1, \text{If}[n==1,1, F[n-1]+F[n-2]]]$$

saves the previously computed values, so at every step there is a single sum to perform.

Exercise 1.1.1. Compare the times that it takes to evaluate

$$F_{30} = 832040 \quad (1.1.2)$$

using both versions of the function $F$. 

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**Factorials and Binomial Coefficients**

A recurrence can also be used to *define* a sequence of numbers. For instance

\[ D_{n+1} = n(D_n + D_{n-1}), \quad n \geq 2 \quad (1.1.3) \]

with \( D_1 = 0, \) \( D_2 = 1 \) defines the *derangement numbers*. See Rosen (2003) for properties of this interesting sequence.

We now give a recursive definition of the factorials.

**Definition 1.1.2.** The *factorial* of \( n \in \mathbb{N} \) is defined by

\[ n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1. \quad (1.1.4) \]

A recursive definition is given by

\[ 1! = 1 \quad (1.1.5) \]
\[ n! = n \times (n - 1)!. \]

The first exercise shows that the recursive definition characterizes \( n! \). This technique will be used throughout the book: in order to prove some identity, you check that both sides satisfy the same recursion and that the initial conditions match.

**Exercise 1.1.2.** Prove that the factorial is the unique solution of the recursion

\[ x_n = n \times x_{n-1} \quad (1.1.6) \]

satisfying the initial condition \( x_1 = 1 \). **Hint.** Let \( y_n = x_n/n! \) and use (1.1.5) to produce a trivial recurrence for \( y_n \).

**Exercise 1.1.3.** Establish the formula

\[ D_n = n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \quad (1.1.7) \]

**Hint.** Check that the right-hand side satisfies the same recurrence as \( D_n \) and then check the initial conditions.

The first values of the sequence \( n! \) are

\[ 1! = 1, \quad 2! = 2, \quad 3! = 6, \quad 4! = 24, \quad (1.1.8) \]

and these grow very fast. For instance

\[ 50! = 3041409320171337804361260816606476884437764156896051 \]
\[ 2000000000000 \]

and 1000! has 2568 digits.
1.2. Prime Numbers and the Factorization of \( n! \)

**Mathematica 1.1.1.** The Mathematica command for \( n! \) is `Factorial[n]`. The reader should check the value 1000! stated above. The number of digits of an integer can be obtained with the Mathematica command `Length[IntegerDigits[n]]`.

The next exercise illustrates the fact that the extension of a function from \( \mathbb{N} \) to \( \mathbb{R} \) sometimes produces unexpected results.

**Exercise 1.1.4.** Use Mathematica to check that

\[
\left( \frac{1}{2} \right)! = \frac{\sqrt{\pi}}{2}.
\]

The exercise is one of the instances in which the factorial is connected to \( \pi \), the fundamental constant of trigonometry. Later we will see that the growth of \( n! \) as \( n \to \infty \) is related to \( e \): the base of natural logarithms. These issues will be discussed in Chapters 5 and 6, respectively. To get a complete explanation for the appearance of \( \pi \), the reader will have to wait until Chapter 10 where we introduce the gamma function.

1.2. Prime Numbers and the Factorization of \( n! \)

In this section we discuss the factorization of \( n! \) into prime factors.

**Definition 1.2.1.** An integer \( n \in \mathbb{N} \) is **prime** if its only divisors are 1 and itself.

The reader is referred to Hardy and Wright (1979) and Ribenboim (1989) for more information about prime numbers. In particular, Ribenbiom’s first chapter contains many proofs of the fact that there are infinitely many primes. Much more information about primes can be found at the site

http://www.utm.edu/research/primes/

The set of prime numbers can be used as building blocks for all integers. This is the content of the **Fundamental Theorem of Arithmetic** stated below.

**Theorem 1.2.1.** Every positive integer can be written as a product of prime numbers. This factorization is unique up to the order of the prime factors.

The proof of this result appears in every introductory book in number theory. For example, see Andrews (1994), page 26, for the standard argument.
Mathematica 1.2.1. The Mathematica command `FactorInteger[n]` gives the complete factorization of the integer $n$. For example `FactorInteger[1001]` gives the prime factorization $1001 = 7 \cdot 11 \cdot 13$. The concept of prime factorization can now be extended to rational numbers by allowing negative exponents. For example

$$\frac{1001}{1003} = 7 \cdot 11 \cdot 13 \cdot 17^{-1} \cdot 59^{-1}.$$  \hspace{1cm} (1.2.1)

The efficient complete factorization of a large integer $n$ is one of the basic questions in computational number theory. The reader should be careful with requesting such a factorization from a symbolic language like Mathematica: the amount of time required can become very large. A safeguard is the command

```
FactorInteger[n, FactorComplete -> False]
```

which computes the small factors of $n$ and leaves a part unfactored. The reader will find in Bressoud and Wagon (2000) more information about these issues.

Definition 1.2.2. Let $p$ be prime and $r \in \mathbb{Q}^+$. Then there are unique integers $a$, $b$, not divisible by $p$, and $m \in \mathbb{Z}$ such that

$$r = \frac{a}{b} \times p^m.$$ \hspace{1cm} (1.2.2)

The $p$-adic valuation of $r$ is defined by

$$v_p(r) = p^{-m}.$$ \hspace{1cm} (1.2.3)

The integer $m$ in (1.2.2) will be called the exponent of $p$ in $m$ and will be denoted by $\mu_p(r)$, that is,

$$v_p(r) = p^{-\mu_p(r)}.$$ \hspace{1cm} (1.2.4)

Extra 1.2.1. The $p$-adic valuation of a rational number gives a new way of measuring its size. In this context, a number is small if it is divisible by a large power of $p$. This is the basic idea behind $p$-adic Analysis. Nice introductions to this topic can be found in Gouvea (1997) and Hardy and Wright (1979).

Exercise 1.2.1. Prove that the valuation $v_p$ satisfies

\begin{align*}
v_p(r_1 r_2) &= v_p(r_1) \times v_p(r_2), \\
v_p(r_1 / r_2) &= v_p(r_1) / v_p(r_2),
\end{align*}
1.2. Prime Numbers and the Factorization of $n!$

and

$$v_p(r_1 + r_2) \leq \text{Max} \left( v_p(r_1), v_p(r_2) \right),$$

with equality unless $v_p(r_1) = v_p(r_2)$.

**Extra 1.2.2.** The $p$-adic numbers have many surprising properties. For instance, a series converges $p$-adically if and only if the general term converges to 0.

**Definition 1.2.3.** The floor of $x \in \mathbb{R}$, denoted by $\lfloor x \rfloor$, is the smallest integer less or equal than $x$. The Mathematica command is `Floor[x].`

We now show that the factorization of $n!$ can be obtained without actually computing its value. This is useful considering that $n!$ grows very fast—for instance $10000!$ has 35660 digits.

**Theorem 1.2.2.** Let $p$ be prime and $n \in \mathbb{N}$. The exponent of $p$ in $n!$ is given by

$$\mu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor. \quad (1.2.5)$$

**Proof.** In the product defining $n!$ one can divide out every multiple of $p$, and there are $\lfloor n/p \rfloor$ such numbers. The remaining factor might still be divisible by $p$ and there are $\lfloor n/p^2 \rfloor$ such terms. Now continue with higher powers of $p$. \[\square\]

Note that the sum in (1.2.5) is finite, ending as soon as $p^k > n$. Also, this sum allows the fast factorization of $n!$. The next exercise illustrates how to do it.

**Exercise 1.2.2.** Count the number of divisions required to obtain

$$50! = 2^{47} \cdot 3^{22} \cdot 5^{12} \cdot 7^8 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47,$$

using (1.2.5).

**Exercise 1.2.3.** Prove that every prime $p \leq n$ appears in the prime factorization of $n!$ and that every prime $p > n/2$ appears to the first power.
There are many expressions for the function \( \mu_p(n) \). We present a proof of one due to Legendre (1830). The result depends on the expansion of an integer in base \( p \). The next exercise describes how to obtain such expansion.

**Exercise 1.2.4.** Let \( n, p \in \mathbb{N} \). Prove that there are integers \( n_0, n_1, \ldots, n_r \) such that
\[
n = n_0 + n_1 p + n_2 p^2 + \cdots + n_r p^r
\]
where \( 0 \leq n_i < p \) for \( 0 \leq i \leq r \). **Hint.** Recall the division algorithm: given \( a, b \in \mathbb{N} \) there are integers \( q, r \), with \( 0 \leq r < b \) such that \( a = qb + r \). To obtain the coefficients \( n_i \) first divide \( n \) by \( p \).

**Theorem 1.2.3.** The exponent of \( p \) in \( n! \) is given by
\[
\mu_p(n!) = \frac{n - s_p(n)}{p - 1},
\]
where \( s_p(n) = n_0 + n_1 + \cdots + n_r \) is the sum of the base-\( p \) digits of \( n \). In particular,
\[
\mu_2(n!) = n - s_2(n).
\]

**Proof.** Write \( n \) in base \( p \) as in (1.2.6). Then
\[
\mu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = (n_1 + n_2 p + \cdots + n_r p^{r-1}) + (n_2 + n_3 p + \cdots + n_r p^{r-2}) + \cdots + n_r,
\]
so that
\[
\mu_p(n!) = n_1 + n_2(1 + p) + n_3(1 + p + p^2) + \cdots + n_r(1 + p + \cdots + p^{r-1})
= \frac{1}{p - 1} \left( n_1(p - 1) + n_2(p^2 - 1) + \cdots + n_r(p^r - 1) \right)
= \frac{n - s_p(n)}{p - 1}.
\]

**Corollary 1.2.1.** The exponent of \( p \) in \( n! \) satisfies
\[
\mu_p(n!) \leq \frac{n - 1}{p - 1},
\]
with equality if and only if \( n \) is a power of \( p \).
1.3. The Role of Symbolic Languages

**Mathematica 1.2.2.** The command `IntegerDigits[n,p]` gives the list of numbers \( n_i \) in Exercise 1.2.4.

**Exercise 1.2.5.** Define

\[
A_1(m) = (2m + 1) \prod_{k=1}^{m} (4k - 1) - \prod_{k=1}^{m} (4k + 1). \quad (1.2.10)
\]

Prove that, for any prime \( p \neq 2 \),

\[
\mu_p(A_1(m)) \geq \mu_p(m!). \quad (1.2.11)
\]

**Hint.** Let 
\[
a_m = \prod_{k=1}^{m} (4k - 1) \quad \text{and} \quad b_m = \prod_{k=1}^{m} (4k + 1)
\]

so that \( a_m \) is the product of the least \( m \) positive integers congruent to 1 modulo 4. Observe that for \( p \geq 3 \) prime and \( k \in \mathbb{N} \), exactly one of the first \( p^k \) positive integers congruent to 3 modulo 4 is divisible by \( p^k \) and the same is true for integers congruent to 1 modulo 4. Conclude that \( A_1(m) \) is divisible by the odd part of \( m! \). For instance,

\[
\frac{A_1(30)}{30!} = \frac{359937762656357183375333}{2^{24}}. \quad (1.2.12)
\]

The products in (1.2.10) will be considered in detail in Section 10.9.

1.3. The Role of Symbolic Languages

In this section we discuss how to use Mathematica to conjecture general closed form formulas. A simple example will illustrate the point.

Exercise 1.2.3 shows that \( n! \) is divisible by a large number of consecutive prime numbers. We now turn this information around to empirically suggest closed-form formulas. Assume that in the middle of a calculation we have obtained the numbers

\[
x_1 = 5356234211328000
\]
\[
x_2 = 102793666719744000
\]
\[
x_3 = 207436908065872000
\]
\[
x_4 = 43913881247588352000
\]
\[
x_5 = 973160803270656000000,
\]

and one hopes that these numbers obey a simple rule. The goal is to obtain a function \( x : \mathbb{N} \to \mathbb{N} \) that interpolates the given values, that is, \( x(i) = x_i \) for \( 1 \leq i \leq 5 \). Naturally this question admits more than one solution, and we will
Factorials and Binomial Coefficients

use Mathematica to find one. The prime factorization of the data is

\[
\begin{align*}
x_1 &= 2^{23} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \\
x_2 &= 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
x_3 &= 2^{18} \cdot 3^{12} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \\
x_4 &= 2^{16} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19^3 \\
x_5 &= 2^{22} \cdot 3^8 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19
\end{align*}
\]

and a moment of reflection reveals that \( x_i \) contains all primes less than \( i + 15 \). This is also true for \( (i + 15)! \), leading to the consideration of \( y_i = x_i / (i + 15)! \). We find that

\[
\begin{align*}
y_1 &= 256 \\
y_2 &= 289 \\
y_3 &= 324 \\
y_4 &= 361 \\
y_5 &= 400,
\end{align*}
\]

so that \( y_i = (i + 15)^2 \). Thus \( x_i = (i + 15)^2 \times (i + 15)! \) is one of the possible rules for \( x_i \). This can be then tested against more data, and if the rule still holds, we have produced the conjecture

\[
z_i = i^2 \times i!, \quad (1.3.1)
\]

where \( z_i = x_{i+15} \).

**Definition 1.3.1.** Given a sequence of numbers \( \{a_k : k \in \mathbb{N}\} \), the function

\[
T(x) = \sum_{k=0}^{\infty} a_k x^k \quad (1.3.2)
\]

is the generating function of the sequence. If the sequence is finite, then we obtain a generating polynomial

\[
T_n(x) = \sum_{k=0}^{n} a_k x^k. \quad (1.3.3)
\]

The generating function is one of the forms in which the sequence \( \{a_k : 0 \leq k \leq n\} \) can be incorporated into an analytic object. Usually this makes it easier to perform calculations with them. Mathematica knows a large number
1.3. The Role of Symbolic Languages

of polynomials, so if \( \{a_k\} \) is part of a known family, then a symbolic search will produce an expression for \( T_n \).

**Exercise 1.3.1.** Obtain a closed-form for the generating function of the Fibonacci numbers. **Hint.** Let \( f(x) = \sum_{n=0}^{\infty} F_n x^n \) be the generating function. Multiply the recurrence (1.1.1) by \( x^n \) and sum from \( n = 1 \) to \( \infty \). In order to manipulate the resulting series observe that

\[
\sum_{n=1}^{\infty} F_{n+1} x^n = \sum_{n=2}^{\infty} F_n x^{n-1} = \frac{1}{x} (f(x) - F_0 - F_1 x).
\]

The answer is \( f(x) = x/(1 - x - x^2) \). The Mathematica command to generate the first \( n \) terms of this is

\[
\text{list}[n] := \text{CoefficientList}[	ext{Normal}[	ext{Series}[x/(1-x-x^2), \{x,0,n-1\}],x]
\]

For example, \( f[10] \) gives \( \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34\} \).

It is often the case that the answer is expressed in terms of more complicated functions. For example, Mathematica evaluates the polynomial

\[
G_n(x) = \sum_{k=0}^{n} k! x^k \tag{1.3.4}
\]

as

\[
G_n(x) = -\frac{e^{-1/x}}{x} \left\{ \Gamma(0, -\frac{1}{x}) + (-1)^n \Gamma(n + 2) \Gamma(-1 - n, -\frac{1}{x}) \right\}, \tag{1.3.5}
\]

where \( e^w \) is the usual **exponential function**, \( \Gamma(x) = \int_{0}^{\infty} t^{x-1} e^{-t} \, dt \) \tag{1.3.6} \)

is the **gamma function**, and

\[
\Gamma(a, x) = \int_{x}^{\infty} t^{a-1} e^{-t} \, dt \tag{1.3.7}
\]

is the **incomplete gamma function**. The exponential function will be discussed in Chapter 5, the gamma function in Chapter 10, and the study of \( \Gamma(a, x) \) is postponed until Volume 2.
1.4. The Binomial Theorem

The goal of this section is to recall the binomial theorem and use it to find closed-form expressions for a class of sums involving binomial coefficients.

**Definition 1.4.1.** The binomial coefficient is

\[
\binom{n}{k} := \frac{n!}{k!(n-k)!}, \quad 0 \leq k \leq n.
\]  \hspace{1cm} (1.4.1)

**Theorem 1.4.1.** Let \( a, b \in \mathbb{R} \) and \( n \in \mathbb{N} \). Then

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\]  \hspace{1cm} (1.4.2)

**Proof.** We use induction. The identity \((a + b)^n = (a + b) \times (a + b)^{n-1}\) and the induction hypothesis yield

\[
(a + b)^n = \sum_{k=1}^{n-1} \binom{n-1}{k} a^{n-k} b^k + \sum_{k=0}^{n-1} \binom{n-1}{k} a^{n-k-1} b^{k+1}
\]

\[
= a^n + \sum_{k=1}^{n-1} \left[ \binom{n-1}{k} + \binom{n-1}{k-1} \right] a^{n-k} b^k + b^n.
\]

The result now follows from the identity

\[
\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},
\]  \hspace{1cm} (1.4.3)

that admits a direct proof using (1.4.1). \(\square\)

**Exercise 1.4.1.** Check the details.

**Note 1.4.1.** The binomial theorem

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]  \hspace{1cm} (1.4.4)

shows that \((1 + x)^n\) is the generating function of the binomial coefficients

\[
\left\{ \binom{n}{k} : 0 \leq k \leq n \right\}.
\]
1.4. The Binomial Theorem

In principle it is difficult to predict if a given sequence will have a simple generating function. Compare (1.3.5) with (1.4.4).

We now present a different proof of the binomial theorem in which we illustrate a general procedure that will be employed throughout the text. The goal is to find and prove an expression for \((a + b)^n\).

a) **Scaling.** The first step is to write

\[(a + b)^n = b^n (1 + x)^n \quad (1.4.5)\]

with \(x = a/b\), so that it suffices to establish (1.4.2) for \(a = 1\).

b) **Guessing the structure.** The second step is to formulate an educated guess on the form of \((1 + x)^n\). Expanding \((1 + x)^n\) (for any specific \(n\)) shows that it is a polynomial in \(x\) of degree \(n\), with positive integer coefficients, that is,

\[(1 + x)^n = \sum_{k=0}^{n} b_{n,k} x^k \quad (1.4.6)\]

for some undetermined \(b_{n,k} \in \mathbb{N}\). Observe that \(x = 0\) yields \(b_{n,0} = 1\).

c) The next step is to find a way to understand the coefficients \(b_{n,k}\).

**Exercise 1.4.2.** Differentiate (1.4.6) to produce the recurrence

\[b_{n,k+1} = \frac{n}{k+1} b_{n-1,k} \quad 0 \leq k \leq n - 1. \quad (1.4.7)\]

Conclude that the numbers \(b_{n,k}\) are determined from (1.4.7) and initial condition \(b_{n,0} = 1\).

We now guess the solution to (1.4.7) by studying the list of coefficients

\[L[n] := \{b_{n,k} : 0 \leq k \leq n\}. \quad (1.4.8)\]

The list \(L[n]\) can be generated symbolically by the command

```mathematica
term[n_,k_]:=If[n==0,1, If[ k==0, 1, n*term[n-1,k-1]/k]]; L[n_]:= Table[ term[n,k], {k,0,n}];
```
that produces a list of the coefficients $b_{n,k}$ from (1.4.7). For instance,

$$
L[0] = \{1\}
$$

$$
L[1] = \{1, 1\}
$$

$$
L[2] = \{1, 2, 1\}
$$

$$
L[3] = \{1, 3, 3, 1\}
$$

$$
L[4] = \{1, 4, 6, 4, 1\}
$$

$$
L[5] = \{1, 5, 10, 10, 5, 1\}
$$

$$
L[6] = \{1, 6, 15, 20, 15, 6, 1\}\). \ (1.4.9)

The reader may now recognize the binomial coefficients (1.4.1) from the list (1.4.9) and conjecture the formula

$$
b_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (1.4.10)
$$

from this data. Naturally this requires \textit{a priori} knowledge of the binomial coefficients. An alternative is to employ the procedure described in Section 1.3 to conjecture (1.4.10) from the data in the list $L[n]$.

The guessing of a closed-form formula from data is sometimes obscured by dealing with small numbers. Mathematica can be used to generate terms in the middle part of $L[100]$. The command

$$
t := \text{Table}[L[100][[i]], \{i,45,49\}]
$$

chooses the elements in positions 45 to 49 in $L[100]$:

$$
L[100][[45]] = 49378235797073715747364762200
$$

$$
L[100][[46]] = 61448471214136179596720592960
$$

$$
L[100][[47]] = 73470998190814997343905056800
$$

$$
L[100][[48]] = 84413487283064039501507937600
$$

$$
L[100][[49]] = 93206558875049876949581100, \ (1.4.11)
$$

and, as before, we examine their prime factorizations to find a pattern.

The prime factorization of $n = L[100][[45]]$ is

$$
n = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 47 \cdot 59 \cdot 61 \cdot 67 \cdot 71 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97,
$$

suggesting the evaluation of $n/97!$. It turns out that this the reciprocal of an
1.4. The Binomial Theorem

integer of 124 digits. Its factorization
\[
\frac{97!}{n} = 2^{91} \cdot 3^{43} \cdot 5^{20} \cdot 7^{13} \cdot 11^8 \cdot 13^7 \cdot 17^5 \cdot 19^4 \cdot 23^3 \cdot 29^2 \cdot 31^2 \cdot 37^2 \cdot 41^2 \\
\cdot 43^2 \cdot 47 \cdot 53
\]
leads to the consideration of
\[
\frac{97!}{n \times 53!} = 2^{42} \cdot 3^{20} \cdot 5^8 \cdot 7^5 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43,
\]
and then of
\[
\frac{97!}{n \times 53! \times 43!} = \frac{2^3 \times 3 \times 11}{5 \times 7}.
\] (1.4.12)
The numbers 97, 53 and 43 now have to be slightly adjusted to produce
\[
n = \frac{100!}{56! \times 44!} = \left( \frac{100}{56} \right).
\] (1.4.13)
Repeating this procedure with the other elements in the list (1.4.11) leads to the conjecture (1.4.10).

Exercise 1.4.3. Use the method described above to suggest an analytic expression for
\[
t_1 = 33422213193503283445319840060700101890113888695441 \\
601636800, \\
t_2 = 4786578310918590163752805570088320851200018220954 \\
9068756000, \\
t_3 = 63273506018045330510555274728827082768779925144537 \\
753208000, \\
t_4 = 77218653725969794800710549093404104300057079699419 \\
429079500.
\]
d) Recurrences. Finally, in order to prove that our guess is correct, define
\[
a_{n,k} := \binom{n}{k}^{-1} b_{n,k}
\] (1.4.14)
and show that (1.4.7) becomes
\[
a_{n,k+1} = a_{n-1,k}, \quad n \geq 1, \quad 0 \leq k \leq n - 1.
\] (1.4.15)
so that \(a_{n,k} \equiv 1\).
Factorials and Binomial Coefficients

Exercise 1.4.4. Check that $b_{n,k} = \binom{n}{k}$ by verifying that $\binom{n}{k}$ satisfies (1.4.7) and that this recurrence admits a unique solution with $b_{n,0} = 1$.

Note 1.4.2. The sequence of binomial coefficients has many interesting properties. We provide some of them in the next exercises. The reader will find much more in

http://www.dms.umontreal.ca/~andrew/Binomial/index.html

Exercise 1.4.5. Prove that the exponent of the central binomial coefficients $C_n = \binom{2n}{n}$ satisfies

$$\mu_p(C_n) = \frac{2s_p(n) - s_p(2n)}{p - 1}. \quad (1.4.16)$$

Hint. Let $n = a_0 + a_1 p + \cdots + a_r p^r$ be the expansion of $n$ in base $p$. Define $\lambda_j$ by $2a_j = \lambda_j p + v_j$, where $0 \leq v_j \leq p - 1$. Check that $\lambda_j$ is either 0 or 1 and confirm the formula

$$\mu_p(C_n) = \sum_{j=0}^r \lambda_j. \quad (1.4.17)$$

In particular $\mu_p(C_n) \leq r + 1$. Check that $C_n$ is always even. When is $C_n/2$ odd?

The binomial theorem yields the evaluation of many finite sums involving binomial coefficients. The discussion on binomial sums presented in this book is nonsystematic; we see them as results of evaluations of some definite integrals. The reader will find in Koepf (1998) and Petkovsek et al. (1996) a more complete analysis of these ideas.

Exercise 1.4.6. Let $n \in \mathbb{N}$.

a) Establish the identities

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} k \binom{n}{k} = 2^{n-1} n$$

$$\sum_{k=0}^{n} k^2 \binom{n}{k} = 2^{n-2} n(n + 1). \quad (1.4.18)$$