

INTRODUCTION TO
FOLIATIONS AND
LIE GROUPOIDS

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Contents

<i>Preface</i>	<i>page</i> vii
<i>Prerequisites</i>	1
1 Foliations	4
1.1 Definition and first examples	5
1.2 Alternative definitions of foliations	9
1.3 Constructions of foliations	14
2 Holonomy and stability	19
2.1 Holonomy	20
2.2 Riemannian foliations	25
2.3 Local Reeb stability	30
2.4 Orbifolds	34
2.5 Global Reeb stability in codimension 1	44
2.6 Thurston's stability theorem	49
3 Two classical theorems	56
3.1 Haefliger's theorem	57
3.1.1 Review of Morse functions	58
3.1.2 Morse functions into codimension 1 foliations	60
3.1.3 Proof of Haefliger's theorem	62
3.2 Novikov's theorem	65
3.2.1 Vanishing cycles	66
3.2.2 Existence of a compact leaf	71
3.2.3 Existence of a Reeb component	77
4 Molino's theory	81
4.1 Transverse parallelizability	82
4.1.1 Homogeneous foliations	82
4.1.2 Transversely parallelizable foliations	86

4.2	Principal bundles	92
4.2.1	Connections on principal bundles	93
4.2.2	Transverse principal bundles	98
4.3	Lie foliations and Molino's theorem	101
4.3.1	Lie foliations	102
4.3.2	The Darboux cover	103
4.3.3	Molino's structure theorem	108
5	Lie groupoids	110
5.1	Definition and first examples	111
5.2	The monodromy and holonomy groupoids	117
5.3	Some general constructions	121
5.4	Equivalence of Lie groupoids	127
5.5	Etale groupoids	134
5.6	Proper groupoids and orbifolds	140
5.7	Principal bundles over Lie groupoids	144
6	Lie algebroids	149
6.1	The Lie algebroid of a Lie groupoid	150
6.2	Definition and examples of Lie algebroids	153
6.3	Lie theory for Lie groupoids	157
6.4	Integrability and developable foliations	160
	<i>References and further reading</i>	166
	<i>Index</i>	170

1

Foliations

Intuitively speaking, a foliation of a manifold M is a decomposition of M into immersed submanifolds, the leaves of the foliation. These leaves are required to be of the same dimension, and to fit together nicely.

Such foliations of manifolds occur naturally in various geometric contexts, for example as solutions of differential equations and integrable systems, and in symplectic geometry. In fact, the concept of a foliation first appeared explicitly in the work of Ehresmann and Reeb, motivated by the question of existence of completely integrable vector fields on three-dimensional manifolds. The theory of foliations has now become a rich and exciting geometric subject by itself, as illustrated by the famous results of Reeb (1952), Haefliger (1956), Novikov (1964), Thurston (1974), Molino (1988), Connes (1994) and many others.

We start this book by describing various equivalent ways of defining foliations. A foliation on a manifold M can be given by a suitable foliation atlas on M , by an integrable subbundle of the tangent bundle of M , or by a locally trivial differential ideal. The equivalence of all these descriptions is a consequence of the Frobenius integrability theorem. We will give several elementary examples of foliations. The simplest example of a foliation on a manifold M is probably the one given by the level sets of a submersion $M \rightarrow N$. In general, a foliation on M is a decomposition of M into leaves which is *locally* given by the fibres of a submersion.

In this chapter we also discuss some first properties of foliations, for instance the property of being orientable or transversely orientable. We show that a transversely orientable foliation of codimension 1 on a manifold M is given by the kernel of a differential 1-form on M , and that this form gives rise to the so-called Godbillon–Vey class. This is a class of degree 3 in the de Rham cohomology of M , which depends only on the foliation and not on the choice of the specific 1-form. Furthermore, we

discuss here several basic methods for constructing foliations. These include the product and pull-back of foliations, the formation of foliations on quotient manifolds, the construction of foliations by ‘suspending’ a diffeomorphism or a group of diffeomorphisms, and foliations associated to actions of Lie groups.

1.1 Definition and first examples

Let M be a smooth manifold of dimension n . A *foliation atlas* of codimension q of M (where $0 \leq q \leq n$) is an atlas

$$(\varphi_i: U_i \longrightarrow \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q)_{i \in I}$$

of M for which the change-of-charts diffeomorphisms φ_{ij} are locally of the form

$$\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$$

with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$. The charts of a foliation atlas are called the *foliation charts*. Thus each U_i is divided into *plaques*, which are the connected components of the submanifolds $\varphi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$, $y \in \mathbb{R}^q$, and the change-of-charts diffeomorphisms preserve this division (Figure 1.1). The plaques globally amalgamate

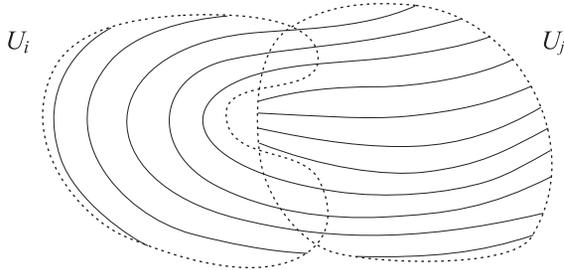


Fig. 1.1. Two foliation charts

into *leaves*, which are smooth manifolds of dimension $n - q$ injectively immersed into M . In other words, two points $x, y \in M$ lie on the same leaf if there exist a sequence of foliation charts U_1, \dots, U_k and a sequence of points $x = p_0, p_1, \dots, p_k = y$ such that p_{j-1} and p_j lie on the same plaque in U_j , for any $1 \leq j \leq k$.

A *foliation of codimension q* of M is a maximal foliation atlas of M of codimension q . Each foliation atlas determines a foliation, since it is

included in a unique maximal foliation atlas. Two foliation atlases define the same foliation of M precisely if they induce the same partition of M into leaves. A (smooth) *foliated manifold* is a pair (M, \mathcal{F}) , where M is a smooth manifold and \mathcal{F} a foliation of M . The *space of leaves* M/\mathcal{F} of a foliated manifold (M, \mathcal{F}) is the quotient space of M , obtained by identifying two points of M if they lie on the same leaf of \mathcal{F} . The dimension of \mathcal{F} is $n - q$. A (smooth) map between foliated manifolds $f: (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a (smooth) map $f: M \rightarrow M'$ which preserves the foliation structure, i.e. which maps leaves of \mathcal{F} into the leaves of \mathcal{F}' .

This is the first definition of a foliation. Instead of smooth foliations one can of course consider C^r -foliations, for any $r \in \{0, 1, \dots, \infty\}$, or (real) analytic foliations. Standard references are Bott (1972), Hector–Hirsch (1981, 1983), Camacho–Neto (1985), Molino (1988) and Tondeur (1988). In the next section we will give several equivalent definitions: in terms of a Haefliger cocycle, in terms of an integrable subbundle of $T(M)$, and in terms of a differential ideal in $\Omega(M)$. But first we give some examples.

Examples 1.1 (1) The space \mathbb{R}^n admits the *trivial* foliation of codimension q , for which the atlas consists of only one chart $\text{id}: \mathbb{R}^n \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$. Of course, any linear bijection $A: \mathbb{R}^n \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$ determines another one whose leaves are the affine subspaces $A^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

(2) Any submersion $f: M \rightarrow N$ defines a foliation $\mathcal{F}(f)$ of M whose leaves are the connected components of the fibres of f . The codimension of $\mathcal{F}(f)$ is equal to the dimension of N . An atlas representing $\mathcal{F}(f)$ is derived from the canonical local form for the submersion f . Foliations associated to the submersions are also called *simple* foliations. The foliations associated to submersions with connected fibres are called *strictly simple*. A simple foliation is strictly simple precisely when its space of leaves is Hausdorff.

(3) (Kronecker foliation of the torus) Let a be an irrational real number, and consider the submersion $s: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $s(x, y) = x - ay$. By (2) we have the foliation $\mathcal{F}(s)$ of \mathbb{R}^2 . Let $f: \mathbb{R}^2 \rightarrow T^2 = S^1 \times S^1$ be the standard covering projection of the torus, i.e. $f(x, y) = (e^{2\pi i x}, e^{2\pi i y})$. The foliation $\mathcal{F}(s)$ induces a foliation \mathcal{F} of T^2 : if φ is a foliation chart for $\mathcal{F}(s)$ such that $f|_{\text{dom}\varphi}$ is injective, then $\varphi \circ (f|_{\text{dom}\varphi})^{-1}$ is a foliation chart for \mathcal{F} . Any leaf of \mathcal{F} is diffeomorphic to \mathbb{R} , and is dense in T^2 (Figure 1.2).

(4) (Foliation of the Möbius band) Let $f: \mathbb{R}^2 \rightarrow M$ be the standard covering projection of the (open) Möbius band: $f(x, y) = f(x', y')$

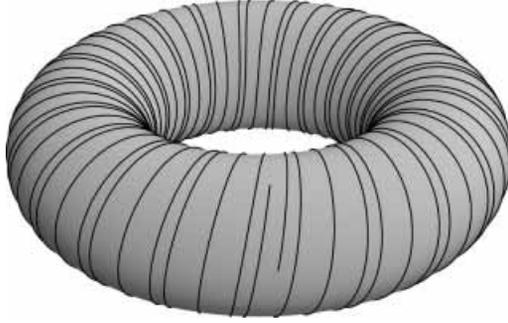


Fig. 1.2. Kronecker foliation of the torus

precisely if $x' - x \in \mathbb{Z}$ and $y' = (-1)^{x'-x}y$. The trivial foliation of codimension 1 of \mathbb{R}^2 induces a foliation \mathcal{F} of M , in the same way as in (3). All the leaves of \mathcal{F} are diffeomorphic to S^1 , and they are wrapping around M twice, except for the ‘middle’ one: this one goes around only once (Figure 1.3).

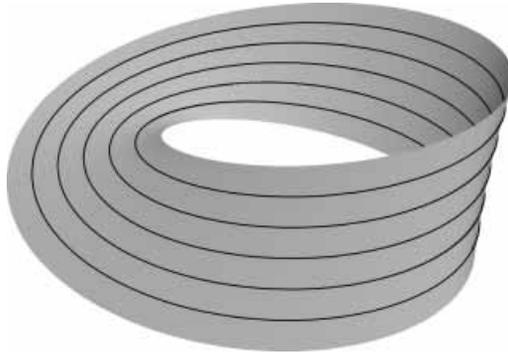


Fig. 1.3. Foliation of the Möbius band

(5) (The Reeb foliation of the solid torus and of S^3) One can also define the notion of a foliation of a manifold with boundary in the obvious way; however, one usually assumes that the leaves of such a foliation behave well near the boundary, by requiring either that they are transversal to the boundary, or that the connected components of the boundary are leaves. An example of the last sort is the Reeb foliation of the solid torus, which is given as follows.

Consider the unit disk $D = \{z \mid z \in \mathbb{C}, |z| \leq 1\}$, and define a submer-

sion $f: \text{Int}(D) \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(z, x) = e^{\frac{1}{1-|z|^2}} - x.$$

So we have the foliation $\mathcal{F}(f)$ of $\text{Int}(D) \times \mathbb{R}$, which can be extended to a foliation of the cylinder $D \times \mathbb{R}$ by adding one new leaf: the boundary $S^1 \times \mathbb{R}$. Now $D \times \mathbb{R}$ is a covering space of the solid torus $X = D \times S^1$ in the canonical way, and the foliation of $D \times \mathbb{R}$ induces a foliation of the solid torus. We will denote this foliation by \mathcal{R} . The boundary torus of this solid torus is a leaf of \mathcal{R} . Any other leaf of \mathcal{R} is diffeomorphic to \mathbb{R}^2 , and has the boundary leaf as its set of adherence points in X . The *Reeb foliation* of X is any foliation \mathcal{F} of X of codimension 1 for which there exists a homeomorphism of X which maps the leaves of \mathcal{F} onto the leaves of \mathcal{R} (Figure 1.4).

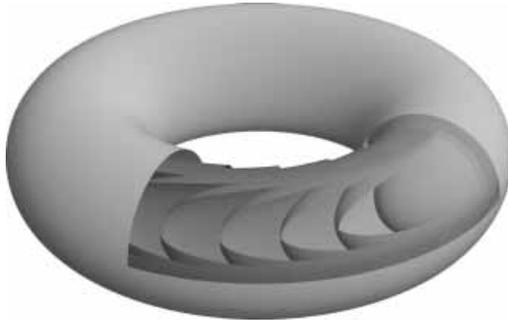


Fig. 1.4. The Reeb foliation of the solid torus

The three-dimensional sphere S^3 can be decomposed into two solid tori glued together along their boundaries, i.e.

$$S^3 \cong X \cup_{\partial X} X.$$

Since ∂X is a leaf of the Reeb foliation of X , we can glue the Reeb foliations of both copies of X along ∂X as well. This can be done so that the obtained foliation of S^3 is smooth. This foliation has a unique compact leaf and is called the *Reeb foliation* of S^3 .

Exercise 1.2 Describe in each of these examples explicitly the space of leaves of the foliation. (You will see that this space often has a very poor structure. Much of foliation theory is concerned with the study of ‘better models’ for the leaf space.)

1.2 Alternative definitions of foliations

A foliation \mathcal{F} of a manifold M can be equivalently described in the following ways (here n is the dimension of M and q the codimension of \mathcal{F}).

- (i) By a foliation atlas $(\varphi_i : U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$ of M for which the change-of-charts diffeomorphisms φ_{ij} are *globally* of the form $\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$ with respect to the decomposition $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$.
- (ii) By an open cover (U_i) of M with submersions $s_i : U_i \rightarrow \mathbb{R}^q$ such that there are diffeomorphisms (necessarily unique)

$$\gamma_{ij} : s_j(U_i \cap U_j) \longrightarrow s_i(U_i \cap U_j)$$

with $\gamma_{ij} \circ s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$. (The diffeomorphisms γ_{ij} satisfy the cocycle condition $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$. This cocycle is called the *Haefliger cocycle* representing \mathcal{F} .)

- (iii) By an *integrable subbundle* E of $T(M)$ of rank $n - q$. (Here integrable (or involutive) means that E is closed under the Lie bracket, i.e. if $X, Y \in \mathfrak{X}(M)$ are sections of E , then the vector field $[X, Y]$ is also a section of E .)
- (iv) By a locally trivial *differential* (graded) *ideal* $\mathcal{J} = \bigoplus_{k=1}^n \mathcal{J}^k$ of rank q in the differential graded algebra $\Omega(M)$. (An ideal \mathcal{J} is locally trivial of rank q if any point of M has an open neighbourhood U such that $\mathcal{J}|_U$ is the ideal in $\Omega(M)|_U$ generated by q linearly independent 1-forms. An ideal \mathcal{J} is differential if $d\mathcal{J} \subset \mathcal{J}$.)

Before we go into details of why these descriptions of the concept of foliation are equivalent, we should point out that the bundle E of (iii) consists of tangent vectors to M which are tangent to the leaves, while a differential k -form is in the ideal \mathcal{J} of (iv) if it vanishes on any k -tuple of vectors which are all tangent to the leaves.

Ad (i): Any foliation atlas $(\varphi_i : U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$ of \mathcal{F} has a refinement which satisfies the condition in (i). To see this, we may first assume that (U_i) is a locally finite cover of M . Next, we may find a locally finite refinement (V_k) of (U_i) such that $V_k \cup V_l$ is contained in some U_i for any non-disjoint V_k and V_l . As any V_k is contained in a U_{i_k} , we may take $\psi_k = \varphi_{i_k}|_{V_k}$. Further we may choose each V_k so small that for any $U_j \supset V_k$, the change-of-charts diffeomorphism $\varphi_j \circ \psi_k^{-1}$ is globally of the form $(g_{jk}(x, y), h_{jk}(y))$, and that h_{jk} is an embedding. This refinement (ψ_k) of (φ_i) is a foliation atlas of M which satisfies the condition in (i).

Ad (ii): If (U_i, s_i, γ_{ij}) is a Haefliger cocycle on M , choose an atlas $(\varphi_k: V_k \rightarrow \mathbb{R}^n)$ so that each V_k is a subset of an U_{i_k} and φ_k renders s_{i_k} in the normal form for a submersion: it is surjective, and there exists a diffeomorphism $\psi_k: s_{i_k}(V_k) \rightarrow \mathbb{R}^q$ such that $\psi_k \circ s_{i_k} = \text{pr}_2 \circ \varphi_k$. This is a foliation atlas of the form in (i): if $(x, y) \in \varphi_k(V_k \cap V_l) \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$, we have

$$\begin{aligned} (\text{pr}_2 \circ \varphi_l \circ \varphi_k^{-1})(x, y) &= (\psi_l \circ s_{i_l} \circ \varphi_k^{-1})(x, y) \\ &= (\psi_l \circ \gamma_{i_l i_k} \circ s_{i_k} \circ \varphi_k^{-1})(x, y) \\ &= (\psi_l \circ \gamma_{i_l i_k} \circ \psi_k)(y) . \end{aligned}$$

Conversely, if $(\varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$ is a foliation atlas of the form in (i), take $s_i = \text{pr}_2 \circ \varphi_i$ and $\gamma_{ij} = h_{ij}$. This gives a Haefliger cocycle on M which represents the same foliation.

Ad (iii): Let us assume that the foliation is given by a foliation atlas $(\varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$. Define a subbundle E of $T(M)$ locally over U_i by

$$E|_{U_i} = \text{Ker}(d(\text{pr}_2 \circ \varphi_i)) ,$$

i.e. by the kernel of the \mathbb{R}^q -valued 1-form $\alpha = d(\text{pr}_2 \circ \varphi_i)$. For any such a 1-form and any vector fields X, Y on U_i we have $2d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$. Since our α is closed, it follows that

$$\alpha([X, Y]) = X(\alpha(Y)) - Y(\alpha(X)) .$$

Using this it is clear that E is an integrable subbundle of $T(M)$ of codimension q .

The bundle E is uniquely determined by the foliation \mathcal{F} : a tangent vector $\xi \in T_x(M)$ is in E precisely if ξ is tangent to the leaf of L through x . The bundle E is called the *tangent bundle* of \mathcal{F} , and is often denoted by $T(\mathcal{F})$. A section of $T(\mathcal{F})$ is called a vector field *tangent* to \mathcal{F} . The Lie algebra $\Gamma(T(\mathcal{F}))$ of sections of $T(\mathcal{F})$ will also be denoted by $\mathfrak{X}(\mathcal{F})$.

Conversely, an integrable subbundle E of codimension q of $T(M)$ can be locally integrated (Frobenius theorem, see Appendix of Camacho–Neto (1985)): for any point $x \in M$ there exist an open neighbourhood $U \subset M$ and a diffeomorphism $\varphi: U \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$ such that $E|_U = \text{Ker}(d(\text{pr}_2 \circ \varphi))$. By using these kinds of diffeomorphisms as foliation charts, one obtains a foliation atlas of the foliation.

Ad (iv): For any subbundle E of $T(M)$, define the (graded) ideal $\mathcal{J} = \bigoplus_{k=1}^n \mathcal{J}^k$ in $\Omega(M)$ as follows: for $\omega \in \Omega^k(M)$,

$$\omega \in \mathcal{J}^k \text{ if and only if}$$

$\omega(X_1, \dots, X_k) = 0$ for any sections X_1, \dots, X_k of E .

Note that \mathcal{J} is locally trivial of rank q , i.e. it is locally generated by q linearly independent 1-forms: Choose a local frame X_1, \dots, X_n of $T(M)|_U$ such that X_1, \dots, X_{n-q} form a frame of $E|_U$. There is the dual frame of differential 1-forms $\omega_1, \dots, \omega_n$ of $T(M)^*|_U$, and the linearly independent 1-forms $\omega_{n-q+1}, \dots, \omega_n$ clearly generate the ideal \mathcal{J} . Conversely, any locally trivial ideal \mathcal{J} of rank q determines a subbundle E of $T(M)$ of rank $n - q$, by the formula above (for $k = 1$).

We claim that under this correspondence,

\mathcal{J} is differential if and only if E is integrable.

In fact, this is immediate from the definition of the exterior derivative:

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= \frac{1}{k+1} \sum_{0 \leq i \leq k} (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \frac{1}{k+1} \sum_{0 \leq j < l \leq k} (-1)^{j+l} \omega([X_j, X_l], X_0, \dots, \hat{X}_j, \dots, \hat{X}_l, \dots, X_k). \end{aligned}$$

REMARKS. (1) Let \mathcal{J} be a locally trivial ideal of rank q in $\Omega(M)$. If \mathcal{J} is differential and locally (over U) generated by $\omega_1, \dots, \omega_q$, then

$$d\omega_i = \sum_{j=1}^q \alpha_{ij} \wedge \omega_j$$

for some $\alpha_{ij} \in \Omega^1(M)|_U$. In particular,

$$d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_q = 0.$$

Conversely, if we have an open cover (U_l) of M such that for any l the restriction $\mathcal{J}|_{U_l}$ is generated by linearly independent 1-forms $\omega_1^l, \dots, \omega_q^l$ satisfying

$$d\omega_i^l \wedge \omega_1^l \wedge \dots \wedge \omega_q^l = 0$$

for any i , then \mathcal{J} is differentiable. Indeed, this implies that

$$d\omega_i^l = \sum_{j=1}^q \alpha_{ij}^l \wedge \omega_j^l$$

for some $\alpha_{ij}^l \in \Omega^1(M)|_{U_l}$; to see this, one should locally complete $\omega_1^l, \dots, \omega_q^l$ to a frame and compute α_{ij}^l locally, and finally obtain α_{ij}^l on U_l using a partition of unity (exercise: fill in the details).

(2) A one-dimensional subbundle E of $T(M)$ (i.e. a *line field*) is clearly integrable, hence any line field on M defines a foliation of M of codimension $n - 1$.

(3) If ω is a nowhere vanishing 1-form on M , it defines a foliation of codimension 1 of M precisely if it is *integrable*, i.e. if

$$d\omega \wedge \omega = 0 .$$

Note that if $\dim M = 2$ then any 1-form ω on M is integrable.

In particular, any closed 1-form on M is integrable. For example, if $\omega = df$ for a smooth map $f: M \rightarrow \mathbb{R}$ without critical points, this gives exactly the foliation given by the submersion f .

Note that if $H_{\text{dR}}^1(M) = 0$ (e.g. if $\pi_1(M)$ is finite) then any closed 1-form ω on M is exact: $\omega = df$. If ω is nowhere vanishing, the function f has no critical points. Hence the foliation given by ω is the foliation given by the submersion f . For example, the Reeb foliation on S^3 , which is clearly not given by a submersion, is hence not given by a closed 1-form.

In general, the integrability condition $d\omega \wedge \omega = 0$ for a nowhere vanishing 1-form ω implies that locally $\omega = gdf$ for a submersion f which locally defines the foliation.

(4) Let (M, \mathcal{F}) and (M', \mathcal{F}') be foliated manifolds. Then a (smooth) map $f: M \rightarrow M'$ preserves the foliation structure (hence it is a map of foliated manifolds) if and only if $df(T(\mathcal{F})) \subset T(\mathcal{F}')$.

Let (M, \mathcal{F}) be a foliated manifold and $T(\mathcal{F})$ the corresponding tangent bundle of \mathcal{F} . We say that \mathcal{F} is *orientable* if the tangent bundle $T(\mathcal{F})$ is orientable, and that \mathcal{F} is *transversely orientable* if its *normal bundle* $N(\mathcal{F}) = T(M)/T(\mathcal{F})$ is orientable. An *orientation* of \mathcal{F} is an orientation of $T(\mathcal{F})$, and a *transverse orientation* of \mathcal{F} is an orientation of $N(\mathcal{F})$.

Exercises 1.3 (1) Show that a foliation \mathcal{F} is transversely orientable if and only if it can be represented by a Haefliger cocycle (U_i, s_i, γ_{ij}) with the property that

$$\det(d\gamma_{ij})_y > 0$$

for any $y \in s_j(U_i \cap U_j)$.

(2) Show that a foliation of codimension 1 is given by a nowhere vanishing integrable 1-form (or a nowhere vanishing vector field) if and only if it is transversely orientable.

(3) Determine which of the foliations in Examples 1.1 are orientable and which are transversely orientable.

(4) Find an example of a foliation of dimension 1 of the Klein bottle, which is neither orientable nor transversely orientable.

Let \mathcal{F} be a transversely orientable foliation of codimension 1 on M . Hence \mathcal{F} is given by an integrable nowhere vanishing differential 1-form ω on M . The form ω is determined uniquely up to the multiplication by a nowhere vanishing smooth function on M .

We have mentioned above that the condition $d\omega \wedge \omega = 0$ implies that $d\omega = \alpha \wedge \omega$. The form α is not uniquely determined, but we shall see that

- (i) $d\alpha \wedge \omega = 0$ and $d(\alpha \wedge d\alpha) = 0$,
- (ii) the class $\text{gv}(\omega) = [\alpha \wedge d\alpha] \in H_{\text{dR}}^3(M)$ is independent of the choice of α , and
- (iii) $\text{gv}(\omega) = \text{gv}(h\omega)$ for any nowhere vanishing smooth function h on M .

It follows the class $\text{gv}(\omega)$ depends only on the foliation \mathcal{F} and not on the particular choice of ω or α . This class is called the *Godbillon–Vey class* of the foliation \mathcal{F} , and is denoted by

$$\text{gv}(\mathcal{F}) \in H_{\text{dR}}^3(M).$$

Let us now prove the properties (i), (ii) and (iii).

(i) Since $d\omega = \alpha \wedge \omega$, we have

$$\begin{aligned} 0 &= dd\omega \\ &= d(\alpha \wedge \omega) \\ &= d\alpha \wedge \omega - \alpha \wedge d\omega \\ &= d\alpha \wedge \omega - \alpha \wedge \alpha \wedge \omega \\ &= d\alpha \wedge \omega. \end{aligned}$$

As before, this implies $d\alpha = \gamma \wedge \omega$ for some 1-form γ . In particular,

$$d(\alpha \wedge d\alpha) = d\alpha \wedge d\alpha = \gamma \wedge \omega \wedge \gamma \wedge \omega = 0.$$

(ii) Let $\alpha' \in \Omega^1(M)$ be another form satisfying $d\omega = \alpha' \wedge \omega$. It follows that $(\alpha' - \alpha) \wedge \omega = 0$, so $\alpha' - \alpha = f\omega$ for a smooth function f on M . Hence

$$\alpha' \wedge d\alpha' = (\alpha + f\omega) \wedge d\alpha' = \alpha \wedge d\alpha' + f\omega \wedge d\alpha'.$$

Note that $\omega \wedge d\alpha' = 0$ by (i). Thus

$$\alpha' \wedge d\alpha' = \alpha \wedge d\alpha'$$

$$\begin{aligned}
&= \alpha \wedge d(\alpha + f\omega) \\
&= \alpha \wedge d\alpha + \alpha \wedge d(f\omega) \\
&= \alpha \wedge d\alpha - d(\alpha \wedge f\omega) .
\end{aligned}$$

The last equation follows from $d(\alpha \wedge f\omega) = d\alpha \wedge f\omega - \alpha \wedge d(f\omega)$ and part (i).

(iii) First we compute

$$\begin{aligned}
d(h\omega) &= dh \wedge \omega + h d\omega \\
&= \frac{1}{h} dh \wedge h\omega + \alpha \wedge h\omega \\
&= (d(\log |h|) + \alpha) \wedge h\omega .
\end{aligned}$$

So with $\alpha'' = d(\log |h|) + \alpha$ we have $\text{gv}(h\omega) = [\alpha'' \wedge d\alpha'']$. But

$$\alpha'' \wedge d\alpha'' = (d(\log |h|) + \alpha) \wedge d\alpha = \alpha \wedge d\alpha + d(\log |h| + \alpha) .$$

1.3 Constructions of foliations

In this section we list some standard constructions of foliations.

Product of foliations. Let (M, \mathcal{F}) and (N, \mathcal{G}) be two foliated manifolds. Then there is the *product foliation* $\mathcal{F} \times \mathcal{G}$ on $M \times N$, which can be constructed as follows. If \mathcal{F} is represented by a Haefliger cocycle (U_i, s_i, γ_{ij}) on M and \mathcal{G} is represented by a Haefliger cocycle $(V_k, s'_k, \gamma'_{kl})$ on N , then $\mathcal{F} \times \mathcal{G}$ is represented by the Haefliger cocycle

$$(U_i \times V_k, s_i \times s'_k, \gamma_{ij} \times \gamma'_{kl})$$

on $M \times N$. We have $\text{codim}(\mathcal{F} \times \mathcal{G}) = \text{codim } \mathcal{F} + \text{codim } \mathcal{G}$ and $T(\mathcal{F} \times \mathcal{G}) = T(\mathcal{F}) \oplus T(\mathcal{G}) \subset T(M) \oplus T(N) = T(M \times N)$.

Pull-back of a foliation. Let $f: N \rightarrow M$ be a smooth map and \mathcal{F} a foliation of M of codimension q . Assume that f is transversal to \mathcal{F} : this means that f is transversal to all the leaves of \mathcal{F} , i.e. for any $x \in N$ we have

$$(df)_x(T_x(N)) + T_{f(x)}(\mathcal{F}) = T_{f(x)}(M) .$$

Then we get a foliation $f^*(\mathcal{F})$ of N as follows.

Suppose that \mathcal{F} is given by the Haefliger cocycle (U_i, s_i, γ_{ij}) on M . Put $V_i = f^{-1}(U_i)$ and $s'_i = s_i \circ f|_{V_i}$. The maps s'_i are submersions. To see this, take any $x \in V_i$. We have to show that

$$(ds'_i)_x = (ds_i)_{f(x)} \circ (df)_x$$

is surjective. But $(ds_i)_{f(x)}$ is surjective and trivial on $T_{f(x)}(\mathcal{F})$, hence it factors through the quotient $w: T_{f(x)}(M) \rightarrow T_{f(x)}(M)/T_{f(x)}(\mathcal{F})$ as a surjective map. Also $w \circ (df)_x$ is surjective since f is transversal to the leaves, and hence $(ds'_i)_x$ is surjective as well. The foliation $f^*(\mathcal{F})$ is now given by the Haefliger cocycle (V_i, s'_i, γ_{ij}) on N . We have $\text{codim } f^*(\mathcal{F}) = \text{codim } \mathcal{F}$ and $T(f^*(\mathcal{F})) = df^{-1}(T(\mathcal{F}))$.

Transverse orientation cover of a foliation. For a foliated manifold (M, \mathcal{F}) put

$$\text{toc}(M, \mathcal{F}) = \{(x, \mathcal{O}) \mid x \in M, \mathcal{O} \text{ orientation of } N_x(\mathcal{F})\}.$$

There is an obvious smooth structure on $\text{toc}(M, \mathcal{F})$ such that the projection $p: \text{toc}(M, \mathcal{F}) \rightarrow M$ is a twofold covering projection, called the *transverse orientation cover* of the foliated manifold (M, \mathcal{F}) . The lift $\text{toc}(\mathcal{F}) = p^*(\mathcal{F})$ of \mathcal{F} to the transverse orientation cover is a transversely orientable foliation.

Orientation cover of a foliation. For any foliated manifold (M, \mathcal{F}) there is also a smooth structure on

$$\text{oc}(M, \mathcal{F}) = \{(x, \mathcal{O}) \mid x \in M, \mathcal{O} \text{ orientation of } T_x(\mathcal{F})\}$$

such that the projection $p: \text{oc}(M, \mathcal{F}) \rightarrow M$ is a twofold covering projection. This covering space is called the *orientation cover* of the foliated manifold (M, \mathcal{F}) . The lift $\text{oc}(\mathcal{F}) = p^*(\mathcal{F})$ of \mathcal{F} to the orientation cover is an orientable foliation.

Exercises 1.4 (1) Show that if \mathcal{F} is a foliation of an orientable manifold M then \mathcal{F} is orientable if and only if \mathcal{F} is transversely orientable.

(2) Find an example of non-orientable foliation of dimension 1 on the torus. What is the orientation cover of that foliation?

(3) By using the (transverse) orientation cover, show that if a compact manifold M carries a foliation of dimension 1 (or of codimension 1) then the Euler characteristic of M is 0. In particular, the only closed surfaces which admit a foliation of dimension 1 are the torus and the Klein bottle.

Quotient foliation. Let (M, \mathcal{F}) be a foliated manifold, and let G be a group acting freely and properly discontinuously by diffeomorphisms on M , so that the quotient manifold M/G is Hausdorff. We assume that the foliation \mathcal{F} is *invariant* under this action of G , which means that any diffeomorphism $g: M \rightarrow M$ in G maps leaves to leaves, or equivalently,

that $dg(T(\mathcal{F})) = T(\mathcal{F})$ for any $g \in G$. Then \mathcal{F} induces a foliation \mathcal{F}/G of M/G in the following way.

First denote by $p: M \rightarrow M/G$ the quotient map, which is a covering projection. Let $(\varphi_i: U_i \rightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q)$ be a foliation atlas of \mathcal{F} . We may assume that $p|_{U_i}$ is injective for any i , by replacing (φ_i) by a refinement if necessary. Then

$$(\varphi_i \circ (p|_{U_i})^{-1}): p(U_i) \longrightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$$

is a foliation atlas representing \mathcal{F}/G . If L is a leaf of \mathcal{F} , then the *isotropy group* $G_L = \{g \in G \mid g(L) = L\}$ of L acts smoothly on L , and the orbit manifold L/G_L can be identified with a leaf of \mathcal{F}/G via the natural immersion of L/G_L into M/G . We have $\text{codim}(\mathcal{F}/G) = \text{codim}(\mathcal{F})$ and $T(\mathcal{F}/G) = dp(T(\mathcal{F}))$. Observe that we already used this construction in Example 1.1 (3).

Suspension of a diffeomorphism. This is another example of a quotient foliation. Let $f: F \rightarrow F$ be a diffeomorphism of a manifold F . The space $\mathbb{R} \times F$ has the obvious foliation of dimension 1, by the leaves $\mathbb{R} \times \{x\}$, $x \in F$. The smooth action of \mathbb{Z} , defined on $\mathbb{R} \times F$ by

$$(k, (t, x)) \mapsto (t + k, f^k(x)),$$

$k \in \mathbb{Z}$, $t \in \mathbb{R}$, $x \in F$, is properly discontinuous and it maps leaves to leaves. Thus we obtain the quotient foliation \mathcal{S}_f on the (Hausdorff) manifold $(\mathbb{R} \times F)/\mathbb{Z} = \mathbb{R} \times_{\mathbb{Z}} F$. The foliated manifold $(\mathbb{R} \times_{\mathbb{Z}} F, \mathcal{S}_f)$ is called the *suspension* of the diffeomorphism f .

Foliation associated to a Lie group action. We first recall some terminology. For a smooth action $G \times M \rightarrow M$, $(g, x) \mapsto gx$, of a Lie group G on a smooth manifold M , the *isotropy* (or *stabilizer*) *subgroup* at $x \in M$ is the subgroup $G_x = \{g \in G \mid gx = x\}$. It is a closed subgroup of G , hence itself a Lie group. The *orbit* of x is $Gx = \{gx \mid g \in G\}$. It can be viewed as a manifold injectively immersed into M , via the immersion $G/G_x \rightarrow M$ with the image Gx .

We say that the action of G on M is *foliated* if $\dim(G_x)$ is a constant function of x . In this case the connected components of the orbits of the action are leaves of a foliation of M . As an integrable subbundle of $T(M)$, this foliation can simply be described in terms of the Lie algebra \mathfrak{g} of G , namely as the image of the derivative of the action, which is a map of vector bundles $\mathfrak{g} \times M \rightarrow T(M)$ of constant rank.

In the case $G = \mathbb{R}$, a smooth \mathbb{R} -action on M is called a *flow* on M .

To such an action $\mu: \mathbb{R} \times M \rightarrow M$ one can associate a vector field X on M by

$$X(x) = \left. \frac{\partial \mu(t, x)}{\partial t} \right|_{t=0}.$$

A non-trivial flow μ is foliated precisely if its associated vector field X vanishes nowhere; in this case the foliation with the orbits of μ is the foliation given by the line field corresponding to X .

Exercise 1.5 Let $R \subset M \times M$ be an equivalence relation on a manifold M . By Godement's theorem (see Serre (1965)), M/R is a smooth manifold whenever R is a submanifold of $M \times M$ and $\text{pr}_2: R \rightarrow M$ is a submersion. Formulate and prove a result which gives sufficient conditions for a foliation \mathcal{F} on M to induce a foliation on M/R .

Flat bundles. The following method of constructing foliations is related to the previous one of quotient foliations, and prepares the reader for the treatment of Reeb stability in Section 2.3.

Let $p: E \rightarrow M$ be a (smooth) fibre bundle over a connected manifold M . Then p is in particular a submersion, and thus defines the foliation $\mathcal{F}(p)$ of E whose leaves are the connected components of the fibres of p , i.e. the leaves are 'vertical'.

Sometimes it is also possible to construct a foliation of E with 'horizontal' leaves, so that p maps each leaf to M as a covering projection. The following construction captures these examples.

Let $G = \pi_1(M, x)$ be the fundamental group of M at a base-point $x \in M$, and let \tilde{M} be the universal cover of M ; or, more generally, suppose that G is any group acting freely and properly discontinuously on a connected manifold \tilde{M} such that $\tilde{M}/G = M$. We will write the action of G on \tilde{M} as a right action. Suppose also that there is a left action by G on a manifold F . Now form the quotient space

$$E = \tilde{M} \times_G F,$$

obtained from the product space $\tilde{M} \times F$ by identifying (yg, z) with (y, gz) for any $y \in \tilde{M}$, $g \in G$ and $z \in F$. Thus E is the orbit space of $\tilde{M} \times F$ with respect to a properly discontinuous action of G . It is also Hausdorff, so it is a manifold. The projection $\text{pr}_1: \tilde{M} \times F \rightarrow \tilde{M}$ induces a submersion

$\pi: E \rightarrow M$, so we have the following commutative diagram:

$$\begin{array}{ccc} \tilde{M} \times F & \longrightarrow & E = \tilde{M} \times_G F \\ \downarrow \text{pr}_1 & & \downarrow \pi \\ \tilde{M} & \longrightarrow & M \end{array}$$

The map $\pi: E \rightarrow M$ has the structure of a fibre bundle over M with fibre F .

Exercise 1.6 Show that the fibre bundles which can be obtained in this way are exactly the fibre bundles with discrete structure group.

The foliation $\mathcal{F}(\text{pr}_2)$ of $\tilde{M} \times F$, which is given by the submersion $\text{pr}_2: \tilde{M} \times F \rightarrow F$, is invariant under the action of G and hence we obtain the quotient foliation $\mathcal{F} = \mathcal{F}(\text{pr}_2)/G$ on E . If $z \in F$ and $G_z \subset G$ is the isotropy group at z of the action by G on F , then the leaf of E obtained from the leaf $\tilde{M} \times \{z\}$ is naturally diffeomorphic to \tilde{M}/G_z , and π restricted to this leaf is the covering $\tilde{M}/G_z \rightarrow M$ of M .

The suspension of a diffeomorphism discussed above is a special case of this construction.