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Corings and Comodules

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Chapter 1

Coalgebras and comodules

Coalgebras and comodules are dualisations of algebras and modules. In this chapter we introduce the basic definitions and study several properties of these notions. The theory of coalgebras over fields and their comodules is well presented in various textbooks (e.g., Sweedler [45], Abe [1], Montgomery [37], Dăscălescu, Năstăsescu and Raianu [14]). Since the tensor product behaves differently over fields and rings, not all the results for coalgebras over fields can be extended to coalgebras over rings. Here we consider base rings from the very beginning, and part of our problems will be to find out which module properties of a coalgebra over a ring are necessary (and sufficient) to ensure the desired properties. In view of the main subject of this book, this chapter can be treated as a preliminary study towards corings. Also for this reason we almost solely concentrate on those properties of coalgebras and comodules that are important from the module theory point of view. The extra care paid to module properties of coalgebras will pay off in Chapter 3.

Throughout, R denotes a commutative and associative ring with a unit.

1 Coalgebras

Intuitively, a coalgebra over a ring can be understood as a dualisation of an algebra over a ring. Coalgebras by themselves are equally fundamental objects as are algebras. Although probably more difficult to understand at the beginning, they are often easier to handle than algebras. Readers with geometric intuition might like to think about algebras as functions on spaces and about coalgebras as objects that encode additional structure of such spaces (for example, group or monoid structure). The main aim of this section is to introduce and give examples of coalgebras and explain the (dual) relationship between algebras and coalgebras.

1.1. Coalgebras. An *R-coalgebra* is an R-module C with R-linear maps

$$\Delta : C \to C \otimes_R C \quad \text{and} \quad \varepsilon : C \to R,$$

called *(coassociative) coproduct* and *counit*, respectively, with the properties

$$(I_C \otimes \Delta) \circ \Delta = (\Delta \otimes I_C) \circ \Delta, \text{ and } (I_C \otimes \varepsilon) \circ \Delta = I_C = (\varepsilon \otimes I_C) \circ \Delta,$$

which can be expressed by commutativity of the diagrams

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_R C \arrow[d, "I_C \otimes \Delta"] \\ C \otimes_R C \arrow[r, "\Delta \otimes I_C"] & C \otimes_R C \otimes_R C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, "I_C"] & C \otimes_R C \arrow[d, "\varepsilon \otimes I_C"] \\ C \otimes_R C \arrow[r, "I_C \otimes \varepsilon"'] & C\,. \end{tikzcd}$$

A coalgebra (C, Δ, ε) is said to be *cocommutative* if $\Delta = \mathsf{tw} \circ \Delta$, where

$$\mathsf{tw} : C \otimes_R C \to C \otimes_R C, \quad a \otimes b \mapsto b \otimes a,$$

is the twist map (cf. 40.1).

1.2. Sweedler's Σ-notation. For an elementwise description of the maps we use the Σ-*notation*, writing for $c \in C$

$$\Delta(c) = \sum_{i=1}^{k} c_i \otimes \tilde{c}_i = \sum c_{\underline{1}} \otimes c_{\underline{2}}.$$

The first version is more precise; the second version, introduced by Sweedler, turnes out to be very handy in explicit calculations. Notice that $c_{\underline{1}}$ and $c_{\underline{2}}$ do not represent single elements but families $c_1, \ldots, c_k$ and $\tilde{c}_1, \ldots, \tilde{c}_k$ of elements of C that are by no means uniquely determined. Properties of $c_{\underline{1}}$ can only be considered in context with $c_{\underline{2}}$. With this notation, the coassociativity of Δ is expressed by

$$\sum \Delta(c_{\underline{1}}) \otimes c_{\underline{2}} = \sum c_{\underline{1}\,\underline{1}} \otimes c_{\underline{1}\,\underline{2}} \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes c_{\underline{2}\,\underline{1}} \otimes c_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}} \otimes \Delta(c_{\underline{2}}),$$

and, hence, it is possible and convenient to shorten the notation by writing

$$\begin{aligned} (\Delta \otimes I_C)\Delta(c) = (I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}}, \\ (I_C \otimes I_C \otimes \Delta)(I_C \otimes \Delta)\Delta(c) &= \sum c_{\underline{1}} \otimes c_{\underline{2}} \otimes c_{\underline{3}} \otimes c_{\underline{4}}, \end{aligned}$$

and so on. The conditions for the counit are described by

$$\sum \varepsilon(c_{\underline{1}})c_{\underline{2}} = c = \sum c_{\underline{1}}\varepsilon(c_{\underline{2}}).$$

Cocommutativity is equivalent to $\sum c_{\underline{1}} \otimes c_{\underline{2}} = \sum c_{\underline{2}} \otimes c_{\underline{1}}$.

R-coalgebras are closely related or dual to algebras. Indeed, the module of R-linear maps from a coalgebra C to any R-algebra is an R-algebra.

1.3. The algebra $\mathrm{Hom}_R(C, A)$. *For any R-linear map $\Delta : C \to C \otimes_R C$ and an R-algebra A,* $\mathrm{Hom}_R(C, A)$ *is an R-algebra by the* convolution product

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \quad \textit{i.e.,} \quad f * g(c) = \sum f(c_{\underline{1}})g(c_{\underline{2}}),$$

for $f, g \in \mathrm{Hom}_R(C, A)$ and $c \in C$. Furthermore, *(1)* Δ *is coassociative if and only if* $\mathrm{Hom}_R(C,A)$ *is an associative R-algebra, for any R-algebra A.*

(2) C *is cocommutative if and only if* $\mathrm{Hom}_R(C,A)$ *is a commutative R-algebra, for any commutative R-algebra A.*

(3) C *has a counit if and only if* $\mathrm{Hom}_R(C,A)$ *has a unit, for all R-algebras A with a unit.*

Proof. (1) Let $f, g, h \in \mathrm{Hom}_R(C,A)$ and consider the R-linear map

$$\tilde{\mu} : A \otimes_R A \otimes_R A \to A, \quad a_1 \otimes a_2 \otimes a_3 \mapsto a_1 a_2 a_3.$$

By definition, the products $(f * g) * h$ and $f * (g * h)$ in $\mathrm{Hom}_R(C,A)$ are the compositions of the maps

$$\begin{tikzcd}
& C \otimes_R C \arrow[dr, "\Delta\otimes I_C"] & & & \\
C \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & C \otimes_R C \otimes_R C \arrow[r, "f\otimes g\otimes h"] & A \otimes_R A \otimes_R A \arrow[r, "\tilde{\mu}"] & A\,. \\
& C \otimes_R C \arrow[ur, "I_C\otimes\Delta"'] & & &
\end{tikzcd}$$

It is obvious that coassociativity of Δ yields associativity of $\mathrm{Hom}_R(C,A)$.

To show the converse, we see from the above diagram that it suffices to prove that, (at least) for one associative algebra A and suitable $f, g, h \in \mathrm{Hom}_R(C,A)$, the composition $\tilde{\mu} \circ (f \otimes g \otimes h)$ is a monomorphism. So let $A = T(C)$, the tensor algebra of the R-module C (cf. 15.12), and $f = g = h$, the canonical mapping $C \to T(C)$. Then $\tilde{\mu} \circ (f \otimes g \otimes h)$ is just the embedding $C \otimes C \otimes C = T_3(C) \to T(C)$.

(2) If C is cocommutative and A is commutative,

$$f * g\,(c) = \sum f(c_{\underline{1}}) g(c_{\underline{2}}) = \sum g(c_{\underline{1}}) f(c_{\underline{2}}) = g * f\,(c),$$

so that $\mathrm{Hom}_R(C,A)$ is commutative. Conversely, assume that $\mathrm{Hom}_R(C,A)$ is commutative for any commutative A. Then

$$\mu \circ (f \otimes g)(\Delta(c)) = \mu \circ (f \otimes g)(\mathsf{tw} \circ \Delta(c)).$$

This implies $\Delta = \mathsf{tw} \circ \Delta$ provided we can find a commutative algebra A and $f, g \in \mathrm{Hom}_R(C,A)$ such that $\mu \circ (f \otimes g) : C \otimes_R C \to A$ is injective. For this take A to be the symmetric algebra $\mathcal{S}(C \oplus C)$ (see 15.13). For f and g we choose the mappings

$$C \to C \oplus C, \quad x \mapsto (x, 0), \quad C \to C \oplus C, \quad x \mapsto (0, x),$$

composed with the canonical embedding $C \oplus C \to \mathcal{S}(C \oplus C)$.

With the canonical isomorphism $h : \mathcal{S}(C) \otimes \mathcal{S}(C) \to \mathcal{S}(C \oplus C)$ (see 15.13) and the embedding $\lambda : C \to \mathcal{S}(C)$, we form $h^{-1} \circ \mu \circ (f \otimes g) = \lambda \otimes \lambda$. Since $\lambda(C)$ is a direct summand of $\mathcal{S}(C)$, we obtain that $\lambda \otimes \lambda$ is injective and so $\mu \circ (f \otimes g)$ is injective.

(3) It is easy to check that the unit in $\mathrm{Hom}_R(C, A)$ is

$$C \xrightarrow{\varepsilon} R \xrightarrow{\iota} A, \quad c \mapsto \varepsilon(c)1_A.$$

For the converse, consider the R-module $A = R \oplus C$ and define a unital R-algebra

$$\mu : A \otimes_R A \to A, \quad (r, a) \otimes (s, b) \mapsto (rs, rb + as).$$

Suppose there is a unit element in $\mathrm{Hom}_R(C, A)$,

$$e : C \to A = R \oplus C, \quad c \mapsto (\varepsilon(c), \lambda(c)),$$

with R-linear maps $\varepsilon : C \to R$, $\lambda : C \to C$. Then, for $f : C \to A$, $c \mapsto (0, c)$, multiplication in $\mathrm{Hom}_R(C, A)$ yields

$$f * e : C \to A, \quad c \mapsto (0, (I_C \otimes \varepsilon) \circ \Delta(c)).$$

By assumption, $f = f * e$ and hence $I_C = (I_C \otimes \varepsilon) \circ \Delta$, one of the conditions for ε to be a counit. Similarly, the other condition is derived from $f = e * f$.

Clearly ε is the unit in $\mathrm{Hom}_R(C, R)$, showing the uniqueness of a counit for C. $\square$

Note in particular that $C^* = \mathrm{Hom}_R(C, R)$ is an algebra with the convolution product known as the *dual* or *convolution algebra* of C.

Notation. From now on, C (usually) will denote a coassociative R-coalgebra (C, Δ, ε), and A will stand for an associative R-algebra with unit (A, μ, ι).

Many properties of coalgebras depend on properties of the base ring R. The base ring can be changed in the following way.

1.4. Scalar extension. *Let C be an R-coalgebra and S an associative commutative R-algebra with unit. Then $C \otimes_R S$ is an S-coalgebra with the coproduct*

$$\tilde{\Delta} : C \otimes_R S \xrightarrow{\Delta \otimes I_S} (C \otimes_R C) \otimes_R S \xrightarrow{\simeq} (C \otimes_R S) \otimes_S (C \otimes_R S)$$

and the counit $\varepsilon \otimes I_S : C \otimes_R S \to S$. If C is cocommutative, then $C \otimes_R S$ is cocommutative.

Proof. By definition, for any $c \otimes s \in C \otimes_R S$,

$$\tilde{\Delta}(c \otimes s) = \sum (c_{\underline{1}} \otimes 1_S) \otimes_S (c_{\underline{2}} \otimes s).$$

It is easily checked that $\tilde{\Delta}$ is coassociative. Moreover,

$$(\varepsilon \otimes I_S \otimes I_{C \otimes_R S}) \circ \tilde{\Delta}(c \otimes s) = \sum \varepsilon(c_{\underline{1}}) c_{\underline{2}} \otimes s = c \otimes s,$$

and similarly $(I_{C \otimes_R S} \otimes \varepsilon \otimes I_S) \circ \tilde{\Delta} = I_{C \otimes_R S}$ is shown.

Obviously cocommutativity of Δ implies cocommutativity of $\tilde{\Delta}$. □

To illustrate the notions introduced above we consider some examples.

1.5. R as a coalgebra. The ring R is (trivially) a coassociative, cocommutative coalgebra with the canonical isomorphism $R \to R \otimes_R R$ as coproduct and the identity map $R \to R$ as counit.

1.6. Free modules as coalgebras. Let F be a free R-module with basis $(f_\lambda)_\Lambda$, Λ any set. Then there is a unique R-linear map

$$\Delta : F \to F \otimes_R F, \quad f_\lambda \mapsto f_\lambda \otimes f_\lambda,$$

defining a coassociative and cocommutative coproduct on F. The counit is provided by the linear map $\varepsilon : F \to R$, $f_\lambda \longmapsto 1$.

1.7. Semigroup coalgebra. Let G be a semigroup. A coproduct and counit on the semigroup ring $R[G]$ can be defined by

$$\Delta_1 : R[G] \to R[G] \otimes_R R[G],\ g \mapsto g \otimes g, \quad \varepsilon_1 : R[G] \to R,\ g \mapsto 1.$$

If G has a unit e, then another possibility is

$$\Delta_2 : R[G] \to R[G] \otimes_R R[G], \quad g \mapsto \begin{cases} e \otimes e & \text{if } g = e, \\ g \otimes e + e \otimes g & \text{if } g \neq e. \end{cases}$$

$$\varepsilon_2 : R[G] \to R, \quad g \mapsto \begin{cases} 1 & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Both Δ_1 and Δ_2 are coassociative and cocommutative.

1.8. Polynomial coalgebra. A coproduct and counit on the polynomial ring $R[X]$ can be defined as algebra homomorphisms by

$$\begin{array}{ll} \Delta_1 : R[X] \to R[X] \otimes_R R[X], & X^i \mapsto X^i \otimes X^i, \\ \varepsilon_1 : R[X] \to R, & X^i \mapsto 1, \quad i = 0, 1, 2, \ldots. \end{array}$$

or else by

$$\begin{array}{ll} \Delta_2 : R[X] \to R[X] \otimes_R R[X], & 1 \mapsto 1,\ X^i \mapsto (X \otimes 1 + 1 \otimes X)^i, \\ \varepsilon_2 : R[X] \to R, & 1 \mapsto 1,\ X^i \mapsto 0, \quad i = 1, 2, \ldots. \end{array}$$

Again, both Δ_1 and Δ_2 are coassociative and cocommutative.

1.9. Coalgebra of a projective module. Let P be a finitely generated projective R-module with dual basis $p_1, \ldots, p_n \in P$ and $\pi_1, \ldots, \pi_n \in P^*$. There is an isomorphism

$$P \otimes_R P^* \to \mathrm{End}_R(P), \quad p \otimes f \mapsto [a \mapsto f(a)p],$$

and on $P^* \otimes_R P$ the coproduct and counit are defined by

$$\Delta : P^* \otimes_R P \to (P^* \otimes_R P) \otimes_R (P^* \otimes_R P), \quad f \otimes p \mapsto \sum_i f \otimes p_i \otimes \pi_i \otimes p,$$

$$\varepsilon : P^* \otimes_R P \to R, \quad f \otimes p \mapsto f(p).$$

By properties of the dual basis,

$$(I_{P \otimes_R P^*} \otimes \varepsilon)\Delta(f \otimes p) = \sum_i f \otimes p_i \pi_i(p) = f \otimes p,$$

showing that ε is a counit, and coassociativity of Δ is proved by the equality

$$(I_{P \otimes_R P^*} \otimes \Delta)\Delta(f \otimes p) = \sum\nolimits_{i,j} f \otimes p_i \otimes \pi_i \otimes p_j \otimes \pi_j \otimes p = (\Delta \otimes I_{P \otimes_R P^*})\Delta(f \otimes p).$$

The dual algebra of $P^* \otimes_R P$ is (anti)isomorphic to $\mathrm{End}_R(P)$ by the bijective maps

$$(P^* \otimes_R P)^* = \mathrm{Hom}_R(P^* \otimes_R P, R) \simeq \mathrm{Hom}_R(P, P^{**}) \simeq \mathrm{End}_R(P),$$

which yield a ring isomorphism or anti-isomorphism, depending from which side the morphisms are acting.

For $P = R$ we obtain $R = R^*$, and $R^* \otimes_R R \simeq R$ is the trivial coalgebra. As a more interesting special case we may consider $P = R^n$. Then $P^* \otimes_R P$ can be identified with the matrix ring $M_n(R)$, and this leads to the

1.10. Matrix coalgebra. Let $\{e_{ij}\}_{1 \le i,j \le n}$ be the canonical R-basis for $M_n(R)$, and define the coproduct and counit

$$\Delta : M_n(R) \to M_n(R) \otimes_R M_n(R), \quad e_{ij} \mapsto \sum\nolimits_k e_{ik} \otimes e_{kj},$$

$$\varepsilon : M_n(R) \to R, \quad e_{ij} \mapsto \delta_{ij}\,.$$

The resulting coalgebra is called the (n,n)-*matrix coalgebra over* R, and we denote it by $M_n^c(R)$.

Notice that the matrix coalgebra may also be considered as a special case of a semigroup coalgebra in 1.7.

From a given coalgebra one can construct the

1.11. Opposite coalgebra. Let $\Delta : C \to C \otimes_R C$ define a coalgebra. Then

$$\Delta^{\mathsf{tw}} : C \xrightarrow{\Delta} C \otimes_R C \xrightarrow{\mathsf{tw}} C \otimes_R C, \quad c \mapsto \sum c_{\underline{2}} \otimes c_{\underline{1}},$$

where tw is the twist map, defines a new coalgebra structure on C known as the *opposite* coalgebra with the same counit. The opposite coalgebra is denoted by C^{cop}. Note that a coalgebra C is cocommutative if and only if C coincides with its opposite coalgebra (i.e., $\Delta = \Delta^{\mathsf{tw}}$).

1.12. Duals of algebras. Let (A, μ, ι) be an R-algebra and assume ${}_RA$ to be finitely generated and projective. Then there is an isomorphism

$$A^* \otimes_R A^* \to (A \otimes_R A)^*, \quad f \otimes g \mapsto [a \otimes b \mapsto f(a)g(b)],$$

and the functor $\mathrm{Hom}_R(-, R) = (-)^*$ yields a coproduct

$$\mu^* : A^* \to (A \otimes_R A)^* \simeq A^* \otimes_R A^*$$

and a counit (as the dual of the unit of A)

$$\varepsilon := \iota^* : A^* \to R, \quad f \mapsto f(1_A).$$

This makes A^* an R-coalgebra that is cocommutative provided μ is commutative. If ${}_RA$ is not finitely generated and projective, the above construction does not work. However, under certain conditions the *finite dual* of A has a coalgebra structure (see 5.7).

Further examples of coalgebras are the tensor algebra 15.12, the symmetric algebra 15.13, and the exterior algebra 15.14 of any R-module, and the enveloping algebra of any Lie algebra.

1.13. Exercises

Let $M_n^c(R)$ be a matrix coalgebra with basis $\{e_{ij}\}_{1 \le i,j \le n}$ (see 1.10). Prove that the dual algebra $M_n^c(R)^*$ is an (n, n)-matrix algebra.

(Hint: Consider the basis of M^* dual to $\{e_{ij}\}_{1 \le i,j \le n}$.)

References. Abuhlail, Gómez-Torrecillas and Wisbauer [50]; Bourbaki [5]; Sweedler [45]; Wisbauer [210].

2 Coalgebra morphisms

To discuss coalgebras formally, one would like to understand not only isolated coalgebras, but also coalgebras in relation to other coalgebras. In a word, one would like to view coalgebras as objects in a category.[1] For this one needs the notion of a *coalgebra morphism*. Such a morphism can be defined as an R-linear map between coalgebras that respects the coalgebra structures (coproducts and counits). The idea behind this definition is of course borrowed from the idea of an algebra morphism as a map respecting the algebra structures. Once such morphisms are introduced, relationships between coalgebras can be studied. In particular, we can introduce the notions of a *subcoalgebra* and a *quotient coalgebra*. These are the topics of the present section.

2.1. Coalgebra morphisms. Given R-coalgebras C and C', an R-linear map $f : C \to C'$ is said to be a *coalgebra morphism* provided the diagrams

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta"'] & C' \arrow[d, "\Delta'"] \\ C \otimes_R C \arrow[r, "f\otimes f"] & C' \otimes_R C' \end{tikzcd}, \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "\varepsilon"'] & C' \arrow[d, "\varepsilon'"] \\ & R \end{tikzcd}$$

are commutative. Explicitly, this means that

$$\Delta' \circ f = (f \otimes f) \circ \Delta, \quad \text{and} \quad \varepsilon' \circ f = \varepsilon,$$

that is, for all $c \in C$,

$$\sum f(c_{\underline{1}}) \otimes f(c_{\underline{2}}) = \sum f(c)_{\underline{1}} \otimes f(c)_{\underline{2}}, \quad \text{and} \quad \varepsilon'(f(c)) = \varepsilon(c).$$

Given an R-coalgebra C and an S-coalgebra D, where S is a commutative ring, a *coalgebra morphism* between C and D is defined as a pair (α, γ) consisting of a ring morphism $\alpha : R \to S$ and an R-linear map $\gamma : C \to D$ such that

$$\gamma' : C \otimes_R S \to D, \quad c \otimes s \mapsto \gamma(c)s,$$

is an S-coalgebra morphism. Here we consider D as an R-module (induced by α) and $C \otimes_R S$ is the scalar extension of C (see 1.4).

As shown in 1.3, for an R-algebra A, the contravariant functor $\mathrm{Hom}_R(-, A)$ turns coalgebras to algebras. It also turns coalgebra morphisms into algebra morphisms.

[1]The reader not familiar with category theory is referred to the Appendix, §38.

2.2. Duals of coalgebra morphisms. *For R-coalgebras C and C', an R-linear map $f : C \to C'$ is a coalgebra morphism if and only if*

$$\mathrm{Hom}(f, A) : \mathrm{Hom}_R(C', A) \to \mathrm{Hom}_R(C, A)$$

is an algebra morphism, for any R-algebra A.

Proof. Let f be a coalgebra morphism. Putting $f^* = \mathrm{Hom}_R(f, A)$, we compute for $g, h \in \mathrm{Hom}_R(C', A)$

$$\begin{aligned} f^*(g * h) &= \mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta \\ &= (g \circ f) * (h \circ f) \;=\; f^*(g) * f^*(h). \end{aligned}$$

To show the converse, assume that f^* is an algebra morphism, that is,

$$\mu \circ (g \otimes h) \circ \Delta' \circ f = \mu \circ (g \otimes h) \circ (f \otimes f) \circ \Delta,$$

for any R-algebra A and $g, h \in \mathrm{Hom}_R(C', A)$. Choose A to be the tensor algebra $T(C)$ of the R-module C and choose g, h to be the canonical embedding $C \to T(C)$ (see 15.12). Then $\mu \circ (g \otimes h)$ is just the embedding $C \otimes_R C \to T_2(C) \to T(C)$, and the above equality implies

$$\Delta' \circ f = (f \otimes f) \circ \Delta,$$

showing that f is a coalgebra morphism. □

2.3. Coideals. The problem of determining which R-submodules of C are kernels of a coalgebra map $f : C \to C'$ is related to the problem of describing the kernel of $f \otimes f$ (in the category of R-modules $\mathbf{M}_R$). If f is surjective, we know that $\mathrm{Ke}\,(f \otimes f)$ is the sum of the canonical images of $\mathrm{Ke}\, f \otimes_R C$ and $C \otimes_R \mathrm{Ke}\, f$ in $C \otimes_R C$ (see 40.15). This suggests the following definition.

The kernel of a surjective coalgebra morphism $f : C \to C'$ is called a *coideal* of C.

2.4. Properties of coideals. *For an R-submodule $K \subset C$ and the canonical projection $p : C \to C/K$, the following are equivalent:*

(a) K is a coideal;

(b) C/K is a coalgebra and p is a coalgebra morphism;

(c) $\Delta(K) \subset \mathrm{Ke}\,(p \otimes p)$ and $\varepsilon(K) = 0$.

If $K \subset C$ is C-pure, then (c) is equivalent to:

(d) $\Delta(K) \subset C \otimes_R K + K \otimes_R C$ and $\varepsilon(K) = 0$.

If (a) holds, then C/K is cocommutative provided C is also.

Proof. (a) $\Leftrightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) There is a commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & C & \xrightarrow{p} & C/K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\bar\Delta} & & \\ 0 & \longrightarrow & \mathrm{Ke}\,(p\otimes p) & \longrightarrow & C\otimes_R C & \xrightarrow{p\otimes p} & C/K\otimes_R C/K & \longrightarrow & 0, \end{array}$$

where commutativity of the right square implies the existence of a morphism $K \to \mathrm{Ke}\,(p\otimes p)$, thus showing $\Delta(K) \subset \mathrm{Ke}\,(p\otimes p)$. For the counit $\bar\varepsilon : C/K \to R$ of C/K, $\bar\varepsilon \circ p = \varepsilon$ and hence $\varepsilon(K) = 0$

(c) $\Rightarrow$ (b) Under the given conditions, the left-hand square in the above diagram is commutative and the cokernel property of p implies the existence of $\bar\Delta$. This makes C/K a coalgebra with the properties required.

(c) $\Leftrightarrow$ (d) If $K \subset C$ is C-pure, $\mathrm{Ke}\,(p\otimes p) = C\otimes_R K + K\otimes_R C$. $\square$

2.5. Factorisation theorem. *Let $f : C \to C'$ be a morphism of R-coalgebras. If $K \subset C$ is a coideal and $K \subset \mathrm{Ke}\, f$, then there is a commutative diagram of coalgebra morphisms*

$$\begin{array}{ccc} C & \xrightarrow{p} & C/K \\ & \searrow^{f} & \downarrow{\scriptstyle\bar f} \\ & & C' . \end{array}$$

Proof. Denote by $\bar f : C/K \to C'$ the R-module factorisation of $f : C \to C'$. It is easy to show that the diagram

$$\begin{array}{ccc} C/K & \xrightarrow{\bar f} & C' \\ {\scriptstyle\bar\Delta}\downarrow & & \downarrow{\scriptstyle\Delta'} \\ C/K\otimes_R C/K & \xrightarrow{\bar f\otimes\bar f} & C'\otimes_R C' \end{array}$$

is commutative. This means that $\bar f$ is a coalgebra morphism. $\square$

2.6. The counit as a coalgebra morphism. *View R as a trivial R-coalgebra as in 1.5. Then, for any R-coalgebra C,*

(1) ε is a coalgebra morphism;

(2) if ε is surjective, then $\mathrm{Ke}\,\varepsilon$ is a coideal.

Proof. (1) Consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{\varepsilon} & R \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\simeq} \\ C\otimes_R C & \xrightarrow{\varepsilon\otimes\varepsilon} & R\otimes_R R \end{array} \qquad \begin{array}{ccc} c & \longmapsto & \varepsilon(c) \\ \downarrow & & \downarrow \\ \sum c_{\underline{1}}\otimes c_{\underline{2}} & \longmapsto & \sum \varepsilon(c_{\underline{1}})\otimes\varepsilon(c_{\underline{2}}) . \end{array}$$

The properties of the counit yield

$$\sum \varepsilon(c_{\underline{1}}) \otimes \varepsilon(c_{\underline{2}}) = \sum \varepsilon(c_{\underline{1}})\varepsilon(c_{\underline{2}}) \otimes 1 = \varepsilon(\sum c_{\underline{1}}\varepsilon(c_{\underline{2}})) \otimes 1 = \varepsilon(c) \otimes 1,$$

so the above diagram is commutative and ε is a coalgebra morphism.

(2)This is clear by (1) and the definition of coideals. □

2.7. Subcoalgebras. An R-submodule D of a coalgebra C is called a *subcoalgebra* provided D has a coalgebra structure such that the inclusion map is a coalgebra morphism.

Notice that a pure R-submodule (see 40.13 for a discussion of purity) $D \subset C$ is a subcoalgebra provided $\Delta_D(D) \subset D \otimes_R D \subset C \otimes_R C$ and $\varepsilon|_D : D \to R$ is a counit for D. Indeed, since D is a pure submodule of C, we obtain

$$\Delta_D(D) = D \otimes_R C \cap C \otimes_R D = D \otimes_R D \subset C \otimes_R C,$$

so that D has a coalgebra structure for which the inclusion is a coalgebra morphism, as required.

From the above observations we obtain:

2.8. Image of coalgebra morphisms. *The image of any coalgebra map $f : C \to C'$ is a subcoalgebra of C'.*

2.9. Remarks. (1) In a general category $\mathbf{A}$, *subobjects* of an object A in $\mathbf{A}$ are defined as *equivalence classes of monomorphisms* $D \to A$. In the definition of subcoalgebras we restrict ourselves to *subsets* (or inclusions) of an object. This will be general enough for our purposes.

(2) The fact that – over arbitrary rings – the tensor product of injective linear maps need not be injective leads to some unexpected phenomena. For example, a submodule D of a coalgebra C can have two distinct coalgebra structures such that, for both of them, the inclusion is a coalgebra map (see Exercise 2.15(3)). It may also happen that, for a submodule V of a coalgebra C, $\Delta(V)$ is contained in the image of the canonical map $V \otimes_R V \to C \otimes_R C$, yet V has no coalgebra structure for which the inclusion $V \to C$ is a coalgebra map (see Exercise 2.15(4)). Another curiosity is that the kernel of a coalgebra morphism $f : C \to C'$ need not be a coideal in case f is not surjective (see Exercise 2.15(5)).

2.10. Coproduct of coalgebras. For a family $\{C_\lambda\}_\Lambda$ of R-coalgebras, put $C = \bigoplus_\Lambda C_\lambda$, the coproduct in $\mathbf{M}_R$, $i_\lambda : C_\lambda \to C$ the canonical inclusions, and consider the R-linear maps

$$C_\lambda \xrightarrow{\Delta_\lambda} C_\lambda \otimes C_\lambda \subset C \otimes C, \quad \varepsilon : C_\lambda \to R.$$

By the properties of coproducts of R-modules there exist unique maps

$$\Delta : C \to C \otimes_R C \text{ with } \Delta \circ i_\lambda = \Delta_\lambda, \quad \varepsilon : C \to R \text{ with } \varepsilon \circ i_\lambda = \varepsilon_\lambda.$$

(C, Δ, ε) is called the *coproduct* (or *direct sum*) of the coalgebras C_λ. It is obvious that the $i_\lambda : C_\lambda \to C$ are coalgebra morphisms.

C is coassociative (cocommutative) if and only if all the C_λ have the corresponding property. This follows – by 1.3 – from the ring isomorphism

$$\mathrm{Hom}_R(C, A) = \mathrm{Hom}_R(\bigoplus_\Lambda C_\lambda, A) \simeq \prod_\Lambda \mathrm{Hom}_R(C_\lambda, A),$$

for any R-algebra A, and the observation that the left-hand side is an associative (commutative) ring if and only if every component in the right-hand side has this property.

Universal property of $C = \bigoplus_\Lambda C_\lambda$. *For a family $\{f_\lambda : C_\lambda \to C'\}_\Lambda$ of coalgebra morphisms there exists a unique coalgebra morphism $f : C \to C'$ such that, for all $\lambda \in \Lambda$, there are commutative diagrams of coalgebra morphisms*

$$\begin{tikzcd} C_\lambda \arrow[r, "i_\lambda"] \arrow[dr, "f_\lambda"'] & C \arrow[d, "f"] \\ & C' . \end{tikzcd}$$

2.11. Direct limits of coalgebras. Let $\{C_\lambda, f_{\lambda\mu}\}_\Lambda$ be a direct family of R-coalgebras (with coalgebra morphisms $f_{\lambda\mu}$) over a directed set Λ. Let $\varinjlim C_\lambda$ denote the direct limit in $\mathbf{M}_R$ with canonical maps $f_\mu : C_\mu \to \varinjlim C_\lambda$. Then the $f_{\lambda\mu} \otimes f_{\lambda\mu} : C_\lambda \otimes C_\lambda \to C_\mu \otimes C_\mu$ form a directed system (in $\mathbf{M}_R$) and there is the following commutative diagram

$$\begin{tikzcd} C_\mu \arrow[r, "\Delta_\mu"] \arrow[d, "f_\mu"'] & C_\mu \otimes C_\mu \arrow[d] \arrow[dr, "f_\mu \otimes f_\mu"] & \\ \varinjlim C_\lambda \arrow[r, "\delta"] & \varinjlim (C_\lambda \otimes C_\lambda) \arrow[r, "\theta"] & \varinjlim C_\lambda \otimes \varinjlim C_\lambda , \end{tikzcd}$$

where the maps δ and θ exist by the universal properties of direct limits. The composition

$$\Delta_{lim} = \theta \circ \delta : \varinjlim C_\lambda \to \varinjlim C_\lambda \otimes \varinjlim C_\lambda$$

turns $\varinjlim C_\lambda$ into a coalgebra such that the canonical map (e.g., [46, 24.2])

$$p : \bigoplus_\Lambda C_\lambda \to \varinjlim C_\lambda$$

is a coalgebra morphism. The counit of $\varinjlim C_\lambda$ is the map ε_{lim} determined by the commutativity of the diagrams

$$\begin{tikzcd} C_\mu \arrow[r, "f_\mu"] \arrow[dr, "\varepsilon_\mu"'] & \varinjlim C_\lambda \arrow[d, "\varepsilon_{lim}"] \\ & R . \end{tikzcd}$$

For any associative R-algebra A,

$$\mathrm{Hom}_R(\varinjlim C_\lambda, A) \simeq \varprojlim \mathrm{Hom}_R(C_\lambda, A) \subset \prod_\Lambda \mathrm{Hom}_R(C_\lambda, A),$$

and from this we conclude – by 1.3 – that the coalgebra $\varinjlim C_\lambda$ is coassociative (cocommutative) whenever all the C_λ are coassociative (cocommutative).

Recall that for the definition of the tensor product of R-algebras A, B, the twist map $\mathsf{tw} : A \otimes_R B \to B \otimes_R A$, $a \otimes b \mapsto b \otimes a$ is needed. It also helps to define the

2.12. Tensor product of coalgebras. Let C and D be two R-coalgebras. Then the composite map

$$C \otimes_R D \xrightarrow{\Delta_C \otimes \Delta_D} (C \otimes_R C) \otimes_R (D \otimes_R D) \xrightarrow{I_C \otimes \mathsf{tw} \otimes I_D} (C \otimes_R D) \otimes_R (C \otimes_R D)$$

defines a coassociative coproduct on $C \otimes_R D$, and with the counits ε_C of C and ε_D of D the map $\varepsilon_C \otimes \varepsilon_D : C \otimes_R D \to R$ is a counit of $C \otimes_R D$. With these maps, $C \otimes_R D$ is called the *tensor product coalgebra* of C and D. Obviously $C \otimes_R D$ is cocommutative provided both C and D are cocommutative.

2.13. Tensor product of coalgebra morphisms. *Let $f : C \to C'$ and $g : D \to D'$ be morphisms of R-coalgebras. The tensor product of f and g yields a coalgebra morphism*

$$f \otimes g : C \otimes_R D \to C' \otimes_R D'.$$

In particular, there are coalgebra morphisms

$$I_C \otimes \varepsilon_D : C \otimes_R D \to C, \quad \varepsilon_C \otimes I_D : C \otimes_R D \to D,$$

which, for any commutative R-algebra A, lead to an algebra morphism

$$\mathrm{Hom}_R(C, A) \otimes_R \mathrm{Hom}_R(D, A) \to \mathrm{Hom}_R(C \otimes_R D, A),$$

$$\xi \otimes \zeta \mapsto (\xi \circ (I_C \otimes \varepsilon_D)) * (\zeta \circ (\varepsilon_C \otimes I_D)),$$

where $*$ denotes the convolution product (cf. 1.3).

Proof. The fact that f and g are coalgebra morphisms implies commutativity of the top square in the diagram

$$\begin{tikzcd}
C \otimes_R D \arrow[r, "f\otimes g"] \arrow[d, "\Delta_C \otimes \Delta_D"'] & C' \otimes_R D' \arrow[d, "\Delta_{C'} \otimes \Delta_{D'}"] \\
C \otimes_R C \otimes_R D \otimes_R D \arrow[r, "f\otimes f\otimes g\otimes g"] \arrow[d, "I_C \otimes \mathsf{tw} \otimes I_D"'] & C' \otimes_R C' \otimes_R D' \otimes_R D' \arrow[d, "I_{C'} \otimes \mathsf{tw} \otimes I_{D'}"] \\
C \otimes_R D \otimes_R C \otimes_R D \arrow[r, "f\otimes g\otimes f\otimes g"] & C' \otimes_R D' \otimes_R C' \otimes_R D'
\end{tikzcd},$$

while the bottom square obviously is commutative by the definitions. Commutativity of the outer rectangle means that $f \otimes g$ is a coalgebra morphism.

By 2.2, the coalgebra morphisms $C \otimes_R D \to C$ and $C \otimes_R D \to D$ yield algebra maps

$$\mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C \otimes_R D, A), \quad \mathrm{Hom}_R(D, A) \to \mathrm{Hom}_R(C \otimes_R D, A)\,,$$

and with the product in $\mathrm{Hom}_R(C \otimes_R D, A)$ we obtain a map

$$\mathrm{Hom}_R(C, A) \times \mathrm{Hom}_R(D, A) \to \mathrm{Hom}_R(C \otimes_R D, A),$$

which is R-linear and hence factorises over $\mathrm{Hom}_R(C, A) \otimes_R \mathrm{Hom}_R(D, A)$. This is in fact an algebra morphism since the image of $\mathrm{Hom}_R(C, A)$ commutes with the image of $\mathrm{Hom}_R(D, A)$ by the equalities

$$\begin{aligned}
&((\xi \circ (I_C \otimes \varepsilon_D)) * (\zeta \circ (\varepsilon_C \otimes I_D)))(c \otimes d) \\
&\quad = \sum \xi \circ (I_C \otimes \varepsilon_D) \otimes \zeta \circ (\varepsilon_C \otimes I_D)(c_{\underline{1}} \otimes d_{\underline{1}} \otimes c_{\underline{2}} \otimes d_{\underline{2}}) \\
&\quad = \sum \xi(c_{\underline{1}}\varepsilon(d_{\underline{1}}))\, \zeta(\varepsilon(c_{\underline{2}})d_{\underline{2}}) \\
&\quad = \sum \xi(c_{\underline{1}}\varepsilon(c_{\underline{2}}), \zeta(\varepsilon(d_{\underline{1}})d_{\underline{2}}) \\
&\quad = \xi(c)\, \zeta(d) = \zeta(d)\, \xi(c) \\
&\quad = ((\zeta \circ (\varepsilon_C \otimes I_D)) * (\xi \circ (I_C \otimes \varepsilon_D)))(c \otimes d),
\end{aligned}$$

where $\xi \in \mathrm{Hom}_R(C, A)$, $\zeta \in \mathrm{Hom}_R(D, A)$ and $c \in C$, $d \in D$. □

To define the comultiplication for the tensor product of two R-coalgebras C, D in 2.12, the twist map $\mathsf{tw} : C \otimes_R D \to D \otimes_R C$ was used. Notice that any such map yields a formal comultiplication on $C \otimes_R D$, whose properties strongly depend on the properties of the map chosen.

2.14. Coalgebra structure on the tensor product. For R-coalgebras $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$, let $\omega : C \otimes_R D \to D \otimes_R C$ be an R-linear map. Explicitly on elements we write $\omega(c \otimes d) = \sum d^\omega \otimes c^\omega$. Denote by $C \ltimes_\omega D$ the R-module $C \otimes_R D$ endowed with the maps

$$\begin{aligned}
\bar{\Delta} = (I_C \otimes \omega \otimes I_D) \circ (\Delta_C \otimes \Delta_D) :&\quad C \otimes_R D \to (C \otimes_R D) \otimes_R (C \otimes_R D), \\
\bar{\varepsilon} = \varepsilon_C \otimes \varepsilon_D :&\quad C \otimes_R D \to R.
\end{aligned}$$

Then $C \ltimes_\omega D$ is an R-coalgebra if and only if the following bow-tie diagram *is commutative (tensor over R):*

$$\begin{tikzcd}
& C\otimes D\otimes D \arrow[ddl, "\omega\otimes I_D"'] & & C\otimes C\otimes D \arrow[ddr, "I_C\otimes\omega"] & \\
& & C\otimes D \arrow[ul, "I_C\otimes\Delta_D"'] \arrow[ur, "\Delta_C\otimes I_D"] \arrow[dl, "I_C\otimes\varepsilon_D"'] \arrow[dr, "\varepsilon_C\otimes I_D"] \arrow[dd, "\omega"] & & \\
D\otimes C\otimes D \arrow[ddr, "I_D\otimes\omega"'] & C & & D & C\otimes D\otimes C \arrow[ddl, "\omega\otimes I_C"] \\
& & D\otimes C \arrow[ul, "\varepsilon_D\otimes I_C"] \arrow[ur, "I_D\otimes\varepsilon_C"'] \arrow[dl, "\Delta_D\otimes I_C"] \arrow[dr, "I_D\otimes\Delta_C"'] & & \\
& D\otimes D\otimes C & & D\otimes C\otimes C\,. &
\end{tikzcd}$$

If this holds, the coalgebra $C \ltimes_\omega D$ is called a *smash coproduct* of C and D.

Proof. Notice that commutativity of the central trapezium (triangles) means

$$(I_D \otimes \varepsilon_C)\,\omega(c \otimes d) = \varepsilon_C(c)d, \quad (\varepsilon_D \otimes I_C)\,\omega(c \otimes d) = \varepsilon_D(d)c.$$

By definition, right counitality of $\bar{\varepsilon}$ requires $(I_{C\otimes_R D} \otimes_R \bar{\varepsilon}) \circ \bar{\Delta} = I_{C\otimes_R D}$, that is,

$$c \otimes d = \sum c_{\underline{1}} \otimes (I_D \otimes \varepsilon_C)\,\omega(c_{\underline{2}} \otimes d_{\underline{1}})\bar{\varepsilon}(d_{\underline{2}}) = \sum c_{\underline{1}} \otimes (I_D \otimes \varepsilon_C)\,\omega(c_{\underline{2}} \otimes d).$$

Applying $\varepsilon_C \otimes I_D$, we obtain the first equality (right triangle) for ω. Similarly, the second equality (left triangle) is derived. A simple computation shows that the two equalities imply counitality.

Coassociativity of $\bar{\Delta}$ means commutativity of the diagram

$$\begin{tikzcd}
C\otimes C\otimes D\otimes D \arrow[r, "I\otimes\omega\otimes I"] & C\otimes D\otimes C\otimes D \arrow[r, "I\otimes I\otimes\Delta\otimes\Delta"] & C\otimes D\otimes C\otimes C\otimes D\otimes D \arrow[d, "I\otimes I\otimes I\otimes\omega\otimes I"] \\
C\otimes D \arrow[u, "\Delta_C\otimes\Delta_D"] \arrow[d, "\Delta_C\otimes\Delta_D"'] & (*) & C\otimes D\otimes C\otimes D\otimes C\otimes D \\
C\otimes C\otimes D\otimes D \arrow[r, "I\otimes\omega\otimes I"] & C\otimes D\otimes C\otimes D \arrow[r, "\Delta\otimes\Delta\otimes I\otimes I"] & C\otimes C\otimes D\otimes D\otimes C\otimes D\,, \arrow[u, "I\otimes\omega\otimes I\otimes I\otimes I"']
\end{tikzcd}$$

which is equivalent to the identity $(*)$

$$\sum c_{\underline{1}}\otimes d_{\underline{1}}{}^{\omega}\otimes c_{\underline{2}}{}^{\omega}{}_{\underline{1}}\otimes d_{\underline{2}\,\underline{1}}{}^{\bar{\omega}}\otimes c_{\underline{2}}{}^{\omega}{}_{\underline{2}}{}^{\bar{\omega}}\otimes d_{\underline{2}\,\underline{2}} = \sum c_{\underline{1}\,\underline{1}}\otimes d_{\underline{1}}{}^{\omega}{}_{\underline{1}}{}^{\bar{\omega}}\otimes c_{\underline{1}\,\underline{2}}{}^{\bar{\omega}}\otimes d_{\underline{1}}{}^{\omega}{}_{\underline{2}}\otimes c_{\underline{2}}{}^{\omega}\otimes d_{\underline{2}}.$$

Applying the map $\varepsilon_C \otimes I_C \otimes I_D \otimes I_D \otimes I_C \otimes \varepsilon_D$ to the last module in the diagram $(*)$ – or to formula $(*)$ – we obtain the commutative diagram and formula

$$\begin{tikzcd}
C \otimes D \otimes D \arrow[r, "\omega\otimes I"] & D \otimes C \otimes D \arrow[r, "I\otimes\Delta_C\otimes I"] & D \otimes C \otimes C \otimes D \arrow[d, "I\otimes I\otimes\omega"] \\
C \otimes D \arrow[u, "I\otimes\Delta_D"] \arrow[d, "\Delta_C\otimes I"'] & (**) & D \otimes C \otimes D \otimes C \\
C \otimes C \otimes D \arrow[r, "I\otimes\omega"] & C \otimes D \otimes C \arrow[r, "I\otimes\Delta_D\otimes I"] & C \otimes D \otimes D \otimes C \arrow[u, "\omega\otimes I\otimes I"'] ,
\end{tikzcd}$$

$$(**) \qquad \sum d_{\underline{1}}{}^{\omega} \otimes c^{\omega}{}_{\underline{1}} \otimes d_{\underline{2}}{}^{\bar\omega} \otimes c^{\omega}{}_{\underline{2}}{}^{\bar\omega} = \sum d^{\omega}{}_{\underline{1}}{}^{\bar\omega} \otimes c_{\underline{1}}{}^{\bar\omega} \otimes d^{\omega}{}_{\underline{2}} \otimes c_{\underline{2}}{}^{\omega}.$$

Now assume formula $(**)$ to be given. Tensoring from the left with the coefficients $c_{\underline{1}}$ and replacing c by the coefficients $c_{\underline{2}}$ we obtain

$$\begin{aligned}
\sum c_{\underline{1}} \otimes d_{\underline{1}}{}^{\omega} \otimes c_{\underline{2}}{}^{\omega}{}_{\underline{1}} \otimes d_{\underline{2}}{}^{\bar\omega} \otimes c_{\underline{2}}{}^{\omega}{}_{\underline{2}}{}^{\bar\omega} &= \sum c_{\underline{1}} \otimes d^{\omega}{}_{\underline{1}}{}^{\bar\omega} \otimes c_{\underline{2}\,\underline{1}}{}^{\bar\omega} \otimes d^{\omega}{}_{\underline{2}} \otimes c_{\underline{2}\,\underline{2}}{}^{\omega} \\
&= \sum c_{\underline{1}\,\underline{1}} \otimes d^{\omega}{}_{\underline{1}}{}^{\bar\omega} \otimes c_{\underline{1}\,\underline{2}}{}^{\bar\omega} \otimes d^{\omega}{}_{\underline{2}} \otimes c_{\underline{2}}{}^{\omega}.
\end{aligned}$$

Now, tensoring with the coefficients $d_{\underline{2}}$ from the right and replacing d by the coefficients $d_{\underline{1}}$ we obtain formula $(*)$. So both conditions $(**)$ and $(*)$ are equivalent to coassociativity of $\bar\Delta$.

Commutativity of the trapezium yields a commutative diagram

$$\begin{tikzcd}
D \otimes C \otimes C \otimes D \arrow[d, "I_D\otimes I_C\otimes\omega"'] \arrow[dr, "I_D\otimes\varepsilon_C\otimes\omega"] & \\
D \otimes C \otimes D \otimes C \arrow[r, "I_D\otimes\varepsilon_C\otimes I_D\otimes I_C"] & D \otimes D \otimes C . \\
C \otimes D \otimes D \otimes C \arrow[u, "\omega\otimes I_D\otimes I_C"] \arrow[ur, "\varepsilon_C\otimes I_D\otimes I_D\otimes I_C"'] &
\end{tikzcd}$$

With this it is easy to see that the diagram $(**)$ reduces to the diagram

$$\begin{tikzcd}
C \otimes D \otimes D \arrow[rr, "\omega\otimes I_D"] & & D \otimes C \otimes D \arrow[d, "I_C\otimes\omega"] \\
C \otimes D \arrow[u, "I_C\otimes\Delta_D"] \arrow[r, "\omega"] & D \otimes C \arrow[r, "\Delta_D\otimes I_C"] & D \otimes D \otimes C,
\end{tikzcd}$$

and a similar argument with $\varepsilon_D \otimes I_C$ yields the diagram

$$\begin{tikzcd}
C \otimes D \arrow[r, "\omega"] \arrow[d, "\Delta_C\otimes I_D"'] & D \otimes C \arrow[r, "I_D\otimes\Delta_C"] & D \otimes C \otimes C \\
C \otimes C \otimes D \arrow[rr, "I_C\otimes\omega"] & & C \otimes D \otimes C . \arrow[u, "\omega\otimes I_C"']
\end{tikzcd}$$

Notice that the two diagrams are the left and right wings of the bow-tie and hence one direction of our assertion is proven.

Commutativity of these diagrams corresponds to the equations

$$\sum d_{\underline{1}}{}^{\omega}\otimes d_{\underline{2}}{}^{\bar\omega}\otimes c^{\omega\bar\omega}=\sum d^{\omega}{}_{\underline{1}}\otimes d^{\omega}{}_{\underline{2}}\otimes c^{\omega},\quad \sum d^{\omega}\otimes c^{\omega}{}_{\underline{1}}\otimes c^{\omega}{}_{\underline{2}}=\sum d^{\omega\bar\omega}\otimes c_{\underline{1}}{}^{\bar\omega}\otimes c_{\underline{2}}{}^{\omega},$$

and – alternatively – these can be obtained by applying $I_D\otimes\varepsilon_C\otimes I_D\otimes I_C$ and $I_D\otimes I_C\otimes\varepsilon_D\otimes I_C$ to equation $(**)$.

For the converse implication assume the bow-tie diagram to be commutative. Then the trapezium is commutative and hence $\bar\varepsilon$ is a counit. Moreover, the above equalities hold. Tensoring the first one with the coefficients $c_{\underline{1}}$ and replacing c by the coefficients $c_{\underline{2}}$ we obtain

$$\sum c_{\underline{1}}\otimes d_{\underline{1}}{}^{\omega}\otimes d_{\underline{2}}{}^{\tilde\omega}\otimes c_{\underline{2}}{}^{\omega\tilde\omega}=\sum c_{\underline{1}}\otimes d^{\omega}{}_{\underline{1}}\otimes d^{\omega}{}_{\underline{2}}\otimes c_{\underline{2}}{}^{\omega}.$$

Applying $\omega\otimes I_D\otimes I_C$ to this equation yields

$$\sum d_{\underline{1}}{}^{\omega\bar\omega}\otimes c_{\underline{1}}{}^{\bar\omega}\otimes d_{\underline{2}}{}^{\tilde\omega}\otimes c_{\underline{2}}{}^{\omega\tilde\omega}=\sum d^{\omega}{}_{\underline{1}}{}^{\bar\omega}\otimes c_{\underline{1}}{}^{\bar\omega}\otimes d^{\omega}{}_{\underline{2}}\otimes c_{\underline{2}}{}^{\omega}.$$

Now, tensoring the second equation with the coefficients $d_{\underline{2}}$ from the right, replacing d by the coefficients $d_{\underline{1}}$ and then applying $I_D\otimes I_C\otimes\omega$ yields

$$\sum d_{\underline{1}}{}^{\omega}\otimes c^{\omega}{}_{\underline{1}}\otimes d_{\underline{2}}{}^{\bar\omega}\otimes c^{\omega}{}_{\underline{2}}{}^{\bar\omega}=\sum d_{\underline{1}}{}^{\omega\bar\omega}\otimes c_{\underline{1}}{}^{\bar\omega}\otimes d_{\underline{2}}{}^{\tilde\omega}\otimes c_{\underline{2}}{}^{\omega\tilde\omega}.$$

Comparing the two equations we obtain $(**)$, proving the coassociativity of $\bar\Delta$. □

Notice that a dual construction and a dual bow-tie diagram apply for the definiton of a general product on the tensor product of two R-algebras A, B by an R-linear map $\omega' : B\otimes_R A\to A\otimes_R B$. A partially dual bow-tie diagram arises in the study of entwining structures between R-algebras and R-coalgebras (cf. 32.1).

2.15. Exercises

(1) Let $g : A\to A'$ be an R-algebra morphism. Prove that, for any R-coalgebra C,

$$\mathrm{Hom}(C,g):\mathrm{Hom}_R(C,A)\to\mathrm{Hom}_R(C,A')$$

is an R-algebra morphism.

(2) Let $f : C\to C'$ be an R-coalgebra morphism. Prove that, if f is bijective then f^{-1} is also a coalgebra morphism.

(3) On the $\mathbb{Z}$-module $C=\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}$ define a coproduct

$$\begin{aligned}\Delta : C\to C\otimes_{\mathbb{Z}} C,\quad &(1,0)\mapsto(1,0)\otimes(1,0),\\ &(0,1)\mapsto(1,0)\otimes(0,1)+(0,1)\otimes(1,0).\end{aligned}$$

On the submodule $D = \mathbb{Z} \oplus 2\mathbb{Z}/4\mathbb{Z} \subset C$ consider the coproducts

$$\begin{aligned} \Delta_1 : D \to D \otimes_{\mathbb{Z}} D, \quad & (1,0) \mapsto (1,0) \otimes (1,0), \\ & (0,2) \mapsto (1,0) \otimes (0,2) + (0,2) \otimes (1,0), \\ \Delta_2 : D \to D \otimes_{\mathbb{Z}} D, \quad & (1,0) \mapsto (1,0) \otimes (1,0), \\ & (0,2) \mapsto (1,0)\otimes(0,2) + (0,2)\otimes(0,2) + (0,2)\otimes(1,0). \end{aligned}$$

Prove that (D, Δ_1) and (D, Δ_2) are not isomorphic but the canonical inclusion $D \to C$ is an algebra morphism for both of them (Nichols and Sweedler [168]).

(4) On the $\mathbb{Z}$-module $C = \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ define a coproduct

$$\begin{aligned} \Delta : C \to C \otimes_{\mathbb{Z}} C, \quad & (1,0) \mapsto 0, \\ & (0,1) \mapsto 4(1,0) \otimes (1,0) \end{aligned}$$

and consider the submodule $V = \mathbb{Z}(2,0) + \mathbb{Z}(0,1) \subset C$. Prove:

(i) Δ is well defined.

(ii) $\Delta(V)$ is contained in the image of $V \otimes_R V \to C \otimes_R C$.

(iii) $\Delta : V \to C \otimes_R C$ has no lifting to $V \otimes_R V$ (check the order of the preimage of $\Delta(0,1)$ in $V \otimes_R V$) (Nichols and Sweedler [168]).

(5) Let $C = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$, denote $c_0 = (1,0,0)$, $c_1 = (0,1,0)$, $c_2 = (0,0,1)$ and define a coproduct

$$\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}, \quad n = 0,1,2.$$

Let $D = \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, denote $d_0 = (1,0)$, $d_1 = (0,1)$ and

$$\Delta(d_0) = d_0 \otimes d_0, \quad \Delta(d_1) = d_0 \otimes d_1 + d_1 \otimes d_0.$$

Prove that the map

$$f : C \to D, \quad c_0 \mapsto d_0,\ c_1 \mapsto 2d_1,\ c_2 \mapsto 0,$$

is a $\mathbb{Z}$-coalgebra morphism and $\Delta(c_2) \notin c_2 \otimes C + C \otimes c_2$ (which implies that $\mathrm{Ke}\, f = \mathbb{Z}c_2$ is not a coideal in C) (Nichols and Sweedler [168]).

(6) Prove that the tensor product of coalgebras yields the product in the category of cocommutative coassociative coalgebras.

(7) Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be R-coalgebras with an R-linear mapping $\omega : C \otimes_R D \to D \otimes_R C$. Denote by $C \ltimes_\omega D$ the R-module $C \otimes_R D$ endowed with the maps $\bar{\Delta}$ and $\bar{\varepsilon}$ as in 2.14. The map ω is said to be *left* or *right conormal* if for any $c \in C$, $d \in D$,

$$(I_D \otimes \varepsilon_C)\, \omega(c \otimes d) = \varepsilon(c)d \ \text{ or } \ (\varepsilon_D \otimes I_C)\, \omega(c \otimes d) = \varepsilon_D(d)c.$$

Prove:

(i) The following are equivalent:

(a) ω is left conormal;

(b) $\varepsilon_C \otimes I_D : C \ltimes_\omega D \to D$ respects the coproduct;

(c) $(I_{C\otimes_R D} \otimes_R \bar{\varepsilon}) \circ \bar{\Delta} = I_{C\otimes_R D}$.

(ii) The following are equivalent:

(a) ω is right conormal;

(b) $I_C \otimes \varepsilon_D : C \ltimes_\omega D \to C$ respects the coproduct;

(c) $(\bar{\varepsilon} \otimes_R I_{C\otimes_R D}) \circ \bar{\Delta} = I_{C\otimes_R D}$.

References. Caenepeel, Militaru and Zhu [9]; Nichols and Sweedler [168]; Sweedler [45]; Wisbauer [210].

3 Comodules

In algebra or ring theory, in addition to an algebra, one would also like to study its modules, that is, Abelian groups on which the algebra acts. Correspondingly, in the coalgebra theory one would like to study R-modules on which an R-coalgebra C *coacts*. Such modules are known as (right) C-comodules, and for any given C they form a category $\mathbf{M}^C$, provided morphisms or *C-comodule maps* are suitably defined. In this section we define the category $\mathbf{M}^C$ and study its properties. The category $\mathbf{M}^C$ in many respects is similar to the category of modules of an algebra, for example, there are Hom-tensor relations, there exist cokernels, and so on, and indeed there is a close relationship between $\mathbf{M}^C$ and the modules of the dual coalgebra C^* (cf. Section 4). On the other hand, however, there are several marked differences between categories of modules and comodules. For example, the category of modules is an Abelian category, while the category of comodules of a coalgebra over a ring might not have kernels (and hence it is not an Abelian category in general). This is an important (lack of) property that is characteristic for coalgebras over rings (if R is a field then $\mathbf{M}^C$ is Abelian), that makes studies of such coalgebras particularly interesting. The ring structure of R and the R-module structure of C play in these studies an important role, which requires careful analysis of R-relative properties of a coalgebra or both C- and R-relative properties of comodules.

As before, R denotes a commutative ring, $\mathbf{M}_R$ the category of R-modules, and C, more precisely (C, Δ, ε), stands for a (coassociative) R-coalgebra (with counit). We first introduce right comodules over C.

3.1. Right C-comodules. For $M \in \mathbf{M}_R$, an R-linear map $\varrho^M : M \to M \otimes_R C$ is called a *right coaction* of C on M or simply a *right C-coaction*. To denote the action of ϱ^M on elements of M we write $\varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}}$.

A C-coaction ϱ^M is said to be *coassociative* and *counital* provided the diagrams

$$\begin{tikzcd} M \arrow[rr, "\varrho^M"] \arrow[d, "\varrho^M"'] & & M \otimes_R C \arrow[d, "I_M \otimes \Delta"] \\ M \otimes_R C \arrow[rr, "\varrho^M \otimes I_C"] & & M \otimes_R C \otimes_R C, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\varrho^M"] \arrow[dr, "I_M"'] & M \otimes_R C \arrow[d, "I_M \otimes \varepsilon"] \\ & M \end{tikzcd}$$

are commutative. Explicitly, this means that, for all $m \in M$,

$$\sum \varrho^M(m_{\underline{0}}) \otimes m_{\underline{1}} = \sum m_{\underline{0}} \otimes \Delta(m_{\underline{1}}), \quad m = \sum m_{\underline{0}} \varepsilon(m_{\underline{1}}).$$

In view of the first of these equations we can shorten the notation and write

$$(I_M \otimes \Delta) \circ \varrho^M(m) = \sum m_{\underline{0}} \otimes m_{\underline{1}} \otimes m_{\underline{2}},$$