The Geometry of Physics
An Introduction
Second Edition

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CHAPTER 1

Manifolds and Vector Fields

Better is the end of a thing than the beginning thereof. Ecclesiastes 7:8

As students we learn differential and integral calculus in the context of euclidean space $\mathbb{R}^n$, but it is necessary to apply calculus to problems involving “curved” spaces. Geodesy and cartography, for example, are devoted to the study of the most familiar curved surface of all, the surface of planet Earth. In discussing maps of the Earth, latitude and longitude serve as “coordinates,” allowing us to use calculus by considering functions on the Earth’s surface (temperature, height above sea level, etc.) as being functions of latitude and longitude. The familiar Mercator’s projection, with its stretching of the polar regions, vividly informs us that these coordinates are badly behaved at the poles: that is, that they are not defined everywhere; they are not “global.” (We shall refer to such coordinates as being “local,” even though they might cover a huge portion of the surface. Precise definitions will be given in Section 1.2.) Of course we may use two sets of “polar” projections to study the Arctic and Antarctic regions. With these three maps we can study the entire surface, provided we know how to relate the Mercator to the polar maps.

We shall soon define a “manifold” to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. A manifold will turn out to be the most general space in which one can use differential and integral calculus with roughly the same facility as in euclidean space. It should be recalled, though, that calculus in $\mathbb{R}^3$ demands special care when curvilinear coordinates are required.

The most familiar manifold is $N$-dimensional euclidean space $\mathbb{R}^N$, that is, the space of ordered $N$ tuples $(x^1, \ldots, x^N)$ of real numbers. Before discussing manifolds in general we shall talk about the more familiar (and less abstract) concept of a submanifold of $\mathbb{R}^N$, generalizing the notions of curve and surface in $\mathbb{R}^3$.

1.1. Submanifolds of Euclidean Space

What is the configuration space of a rigid body fixed at one point of $\mathbb{R}^n$?
Euclidean space, $\mathbb{R}^N$, is endowed with a global coordinate system $(x^1, \ldots, x^N)$ and is the most important example of a manifold.

In our familiar $\mathbb{R}^3$, with coordinates $(x, y, z)$, a locus $z = F(x, y)$ describes a (2-dimensional) surface, whereas a locus of the form $y = G(x), z = H(x)$, describes a (1-dimensional) curve. We shall need to consider higher-dimensional versions of these important notions.

A subset $M = M^n \subset \mathbb{R}^{n+r}$ is said to be an $n$-dimensional submanifold of $\mathbb{R}^{n+r}$, *if locally* $M$ can be described by giving $r$ of the coordinates differentiably in terms of the $n$ remaining ones. This means that given $p \in M$, a neighborhood of $p$ on $M$ can be described in some coordinate system $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^r)$ of $\mathbb{R}^{n+r}$ by $r$ differentiable functions

$$y^\alpha = f^\alpha(x^1, \ldots, x^n), \quad \alpha = 1, \ldots, r$$

We abbreviate this by $y = f(x)$, or even $y = y(x)$. We say that $x^1, \ldots, x^n$ are local (curvilinear) coordinates for $M$ near $p$.

Examples:

(i) $y^1 = f(x^1, \ldots, x^n)$ describes an $n$-dimensional submanifold of $\mathbb{R}^{n+1}$.

![Figure 1.1](image_url)

In Figure 1.1 we have drawn a portion of the submanifold $M$. This $M$ is the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $M = \{(x, y) \in \mathbb{R}^{n+1} \mid y = f(x)\}$. When $n = 1$, $M$ is a curve; while if $n = 2$, it is a surface.

(ii) The *unit sphere* $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Points in the northern hemisphere can be described by $z = F(x, y) = (1 - x^2 - y^2)^{1/2}$ and this function is differentiable everywhere except at the equator $x^2 + y^2 = 1$. Thus $x$ and $y$ are local coordinates for the northern hemisphere except at the equator. For points on the equator one can solve for $x$ or $y$ in terms of the others. If we have solved for $x$ then $y$ and $z$ are the two local coordinates. For points in the southern hemisphere one can use the negative square
root for \( z \). The unit sphere in \( \mathbb{R}^3 \) is a 2-dimensional submanifold of \( \mathbb{R}^3 \). We note that we have not been able to describe the entire sphere by expressing one of the coordinates, say \( z \), in terms of the two remaining ones, \( z = F(x, y) \). We settle for local coordinates.

More generally, given \( r \) functions \( F^\alpha(x_1, \ldots, x_n, y_1, \ldots, y_r) \) of \( n + r \) variables, we may consider the locus \( M^n \subset \mathbb{R}^{n+r} \) defined by the equations

\[ F^\alpha(x, y) = c^\alpha, \quad (c^1, \ldots, c^r) \text{ constants} \]

If the Jacobian determinant

\[ \begin{vmatrix} \frac{\partial F^1}{\partial x^1}, \ldots, \frac{\partial F^r}{\partial x^1} \\ \frac{\partial F^1}{\partial y^1}, \ldots, \frac{\partial F^r}{\partial y^1} \end{vmatrix} (x_0, y_0) \]

at \((x_0, y_0) \in M\) of the locus is not 0, the implicit function theorem assures us that locally, near \((x_0, y_0)\), we may solve \( F^\alpha(x, y) = c^\alpha, \alpha = 1, \ldots, r \), for the \( y^\alpha \)'s in terms of the \( x^i \)'s

\[ y^\alpha = f^\alpha(x^1, \ldots, x^n) \]

We may say that “a portion of \( M^n \) near \((x_0, y_0)\) is a submanifold of \( \mathbb{R}^{n+r} \).” If the Jacobian \( \neq 0 \) at all points of the locus, then the entire \( M^n \) is a submanifold.

Recall that the Jacobian condition arises as follows. If \( F^\alpha(x, y) = c^\alpha \) can be solved for the \( y^\beta \)'s differentiably in terms of the \( x^i \)'s, \( y^\beta = y^\beta(x) \), then if, for fixed \( i \), we differentiate the identity \( F^\alpha(x, y(x)) = c^\alpha \) with respect to \( x^i \), we get

\[ \frac{\partial F^\alpha}{\partial x^i} + \sum_{\beta} \left( \frac{\partial F^\alpha}{\partial y^\beta} \right) \frac{\partial y^\beta}{\partial x^i} = 0 \]

and

\[ \frac{\partial y^\beta}{\partial x^i} = -\sum_{\alpha} \left( \left[ \frac{\partial F^\alpha}{\partial y} \right]^{-1} \right)^\beta_{\alpha} \frac{\partial F^\alpha}{\partial x^i} \]

provided the subdeterminant \( \partial(F^1, \ldots, F^r)/\partial(y^1, \ldots, y^r) \) is not zero. (Here \( ([\partial F/\partial y]^{-1})^\beta_{\alpha} \) is the \( \beta\alpha \) entry of the inverse to the matrix \( \partial F/\partial y \); we shall use the convention that for matrix indices, the index to the left always is the row index, whether it is up or down.) This suggests that if the indicated Jacobian is nonzero then we might indeed be able to solve for the \( y^\beta \)'s in terms of the \( x^i \)'s, and the implicit function theorem confirms this. The (nontrivial) proof of the implicit function theorem can be found in most books on real analysis.

Still more generally, suppose that we have \( r \) functions of \( n+r \) variables, \( F^\alpha(x^1, \ldots, x^{n+r}) \). Consider the locus \( F^\alpha(x) = c^\alpha \). Suppose that at each point \( x_0 \) of the locus the Jacobian matrix

\[ \left( \frac{\partial F^\alpha}{\partial x^i} \right) \quad \alpha = 1, \ldots, r \quad i = 1, \ldots, n + r \]

has rank \( r \). Then the equations \( F^\alpha = c^\alpha \) define an \( n \)-dimensional submanifold of \( \mathbb{R}^{n+r} \), since we may locally solve for \( r \) of the coordinates in terms of the remaining \( n \).
In Figure 1.2, two surfaces $F = 0$ and $G = 0$ in $\mathbb{R}^3$ intersect to yield a curve $M$.

The simplest case is one function $F$ of $N$ variables $(x^1, \ldots, x^N)$. If at each point of the locus $F = c$ there is always at least one partial derivative that does not vanish, then the Jacobian (row) matrix $[\partial F/\partial x^1, \partial F/\partial x^2, \ldots, \partial F/\partial x^N]$ has rank 1 and we may conclude that this locus is indeed an $(N - 1)$-dimensional submanifold of $\mathbb{R}^N$. This criterion is easily verified, for example, in the case of the 2-sphere $F(x, y, z) = x^2 + y^2 + z^2 = 1$ of Example (ii). The column version of this row matrix is called in calculus the gradient vector of $F$. In $\mathbb{R}^3$ this vector

$$
\begin{bmatrix}
\frac{\partial F}{\partial x} \\
\frac{\partial F}{\partial y} \\
\frac{\partial F}{\partial z}
\end{bmatrix}
$$

is orthogonal to the locus $F = 0$, and we may conclude, for example, that if this gradient vector has a nontrivial component in the $z$ direction at a point of $F = 0$, then locally we can solve for $z = z(x, y)$.

A submanifold of dimension $(N - 1)$ in $\mathbb{R}^N$, that is, of “codimension” 1, is called a hypersurface.

(iii) The $x$ axis of the $xy$ plane $\mathbb{R}^2$ can be described (perversely) as the locus of the quadratic $F(x, y) := y^2 = 0$. Both partial derivatives vanish on the locus, the $x$ axis, and our criteria would not allow us to say that the $x$ axis is a 1-dimensional submanifold of $\mathbb{R}^2$. Of course the $x$ axis is a submanifold; we should have used the usual description $G(x, y) := y = 0$. Our Jacobian criteria are sufficient conditions, not necessary ones.

(iv) The locus $F(x, y) := xy = 0$ in $\mathbb{R}^2$, consisting of the union of the $x$ and $y$ axes, is not a 1-dimensional submanifold of $\mathbb{R}^2$. It seems “clear” (and can be proved) that in a neighborhood of the intersection of the two lines we are not going to be able to describe the locus in the form of $y = f(x)$ or $x = g(y)$, where $f, g$ are differentiable functions. The best we can say is that this locus with the origin removed is a 1-dimensional submanifold.
1.1b. The Geometry of Jacobian Matrices: The “Differential”

The tangent space to $\mathbb{R}^n$ at the point $x$, written here as $\mathbb{R}^n_x$, is by definition the vector space of all vectors in $\mathbb{R}^n$ based at $x$ (i.e., it is a copy of $\mathbb{R}^n$ with origin shifted to $x$).

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^r$ be coordinates for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. Let $F : \mathbb{R}^n \to \mathbb{R}^r$ be a smooth map. (“Smooth” ordinarily means infinitely differentiable. For our purposes, however, it will mean differentiable at least as many times as is necessary in the present context. For example, if $F$ is once continuously differentiable, we may use the chain rule in the argument to follow.) In coordinates, $F$ is described by giving $r$ functions of $n$ variables

$$y^\alpha = F^\alpha(x) \quad \alpha = 1, \ldots, r$$

or simply $y = F(x)$. We will frequently use the more dangerous notation $y = y(x)$.

Let $y_0 = F(x_0)$; the Jacobian matrix $(\partial y^\alpha / \partial x^i)(x_0)$ has the following significance.

Let $v$ be a tangent vector to $\mathbb{R}^n$ at $x_0$. Take any smooth curve $x(t)$ such that $x(0) = x_0$ and $\dot{x}(0) := (dx/dt)(0) = v$, for example, the straight line $x(t) = x_0 + tv$. The image of this curve

$$y(t) = F(x(t))$$

has a tangent vector $w$ at $y_0$ given by the chain rule

$$w^\alpha = \dot{y}^\alpha(0) = \sum_{i=1}^n \frac{\partial y^\alpha}{\partial x^i}(x_0) \dot{x}^i(0) = \sum_{i=1}^n \left( \frac{\partial y^\alpha}{\partial x^i} \right)(x_0) v^i$$

The assignment $v \mapsto w$ is, from this expression, independent of the curve $x(t)$ chosen, and defines a linear transformation, the differential of $F$ at $x_0$

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0} \quad F_*(v) = w$$ (1.1)
whose matrix is simply the Jacobian matrix \((\partial y^\alpha / \partial x^i)(x_0)\). This interpretation of the Jacobian matrix, as a linear transformation sending tangents to curves into tangents to the image curves under \(F\), can sometimes be used to replace the direct computation of matrices. This philosophy will be illustrated in Section 1.1d.

1.1c. The Main Theorem on Submanifolds of \(\mathbb{R}^N\)

The main theorem is a geometric interpretation of what we have discussed. Note that the statement “\(F\) has rank \(r\) at \(x_0\),” that is, \([\partial y^\alpha / \partial x^i](x_0)\) has rank \(r\), is geometrically the statement that the differential

\[ F_*: \mathbb{R}^n_{x_0} \rightarrow \mathbb{R}^r_{y_0=F(x_0)} \]

is onto or “surjective”; that is, given any vector \(w\) at \(y_0\) there is at least one vector \(v\) at \(x_0\) such that \(F_*(v) = w\). We then have

**Theorem (1.2):** Let \(F: \mathbb{R}^{r+n} \rightarrow \mathbb{R}^r\) and suppose that the locus

\[ F^{-1}(y_0) := \{x \in \mathbb{R}^{r+n} \mid F(x) = y_0\} \]

is not empty. Suppose further that for all \(x_0 \in F^{-1}(y_0)\)

\[ F_*: \mathbb{R}^{n+r}_{x_0} \rightarrow \mathbb{R}^r_{y_0} \]

is onto. Then \(F^{-1}(y_0)\) is an \(n\)-dimensional submanifold of \(\mathbb{R}^{n+r}\).