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Birational Geometry of Algebraic Varieties

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Introduction

From the beginnings of algebraic geometry it has been understood that birationally equivalent varieties have many properties in common. Thus it is natural to attempt to find in each birational equivalence class a variety which is simplest in some sense, and then study these varieties in detail.

Each irreducible curve is birational to a unique smooth projective curve, thus the investigation of smooth projective curves is equivalent to the study of all curves up to birational equivalence.

For surfaces the situation is more complicated. Each irreducible surface is birational to infinitely many smooth projective surfaces. The theory of minimal models of surfaces, developed by the Italian algebraic geometers at the beginning of the twentieth century, aims to choose a unique smooth projective surface from each birational equivalence class. The recipe is quite simple. If a smooth projective surface contains a smooth rational curve with self-intersection -1, then it can be contracted to a point and we obtain another smooth projective surface. Repeating this procedure as many times as possible, we usually obtain a unique 'minimal model'. In a few cases we obtain a model that is not unique, but these cases can be described very explicitly.

A search for a higher dimensional analogue of this method started quite late. One reason is that some examples indicated that a similar approach fails in higher dimensions.

The works of Reid and Mori in the early 1980s raised the possibility that a higher dimensional theory of minimal models may be possible if we allow not just smooth varieties but also varieties with certain mild singularities. This approach is called the Minimal Model Program or Mori's Program. After many contributions by Benveniste, Kawamata, Kollár, Reid, Shokurov, Tsunoda, Viehweg and others, the program was completed in dimension three by Mori in 1988.

Since then this program has grown into a method which can be applied successfully to many problems in algebraic geometry.

The aim of this book is to provide an introduction to the techniques and ideas of the minimal model program.

Chapter 1 gives an introduction to the whole program through a geometric approach. Most of these results are not used later, but they provide a useful conceptual foundation.

Chapter 2 is still introductory, discussing some aspects of singularities and the relevant generalizations of the Kodaira Vanishing Theorem.

The first major part of the program, the Cone Theorem, is proved in Chapter 3. These results work in all dimensions.

The rest of the book is essentially devoted to the study of 3–dimensional flips and flops. Flips and flops are new types of birational transformations which first appear in dimension 3. Most major differences between the theory of surfaces and 3–folds can be traced back to flips and flops.

Chapter 4 is devoted to the classification of certain surface singularities. These results are needed in further work on the 3–dimensional theory.

The singularities appearing in the course of the minimal model program are investigated in Chapter 5. The results are again rather complete in all dimensions.

Flops are studied in Chapter 6. Flops are easier to understand than flips, and, at least in dimension 3, their description is rather satisfactory.

Chapter 7 is devoted to 3–dimensional flips. The general theory is still too complicated and long to be included in a textbook, thus we restrict ourselves to the study of a special class, the so-called semi-stable flips. We have succeeded in simplifying the proofs in this case considerably. Semi-stable flips appear naturally in many contexts, and they are sufficient for several of the applications.

A more detailed description of the contents of each chapter is given at its beginning.

Sections 4.5 and 5.5 are each a side direction, rather than being part of the main line of arguments. In each case we felt that the available references do not adequately cover some results we need, and that our presentation may be of interest to the reader.

Prerequisites

We assume that the reader is familiar with basic algebraic geometry, at the level of [Har77].

There are a few other results that we use without proof.

In the proof of (1.10) we need an estimate for the dimension of the deformation space of a morphism. This result, whose proof is rather technical, is fundamental for much of sections 1.1 and 1.2. These theorems are, however, not used in subsequent sections.

In section 1.5 we recall the basic properties of intersection numbers of divisors and a weak form of Riemann–Roch that we need frequently.

In section 2.4 we state and use the basic comparison theorem of algebraic and analytic cohomologies and also a special case of the Hodge decomposition of the singular cohomology.

In all these cases we need only the stated results, not the techniques involved in their proofs.

A few times we need the Leray spectral sequence (see [God58, 4.17], [HS71, VIII.9] or [Bre97, IV.6] for proofs):

Theorem 0.1. *Let $f : X \to Y$ be a morphism of schemes and F a quasi-coherent sheaf on X. Then there is a spectral sequence*

$$E_2^{i,j} = H^i(Y, R^j f_* F) \Rightarrow H^{i+j}(X, F).$$

We also use resolution of singularities from [Hir64] on many occasions. We need two versions of this result as follows:

Theorem 0.2. *Let X be an irreducible reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighbourhood of a compact set of an irreducible reduced analytic space) and $I \subset \mathcal{O}_X$ a coherent sheaf of ideals defining a closed subscheme (or subspace) Z. Then there are a smooth variety (or analytic space) Y and a projective morphism $f : Y \to X$ such that*

(1) *f is an isomorphism over $X \setminus (\operatorname{Sing}(X) \cup \operatorname{Supp} Z)$,*
(2) *$f^* I \subset \mathcal{O}_Y$ is an invertible sheaf $\mathcal{O}_Y(-D)$ and*
(3) *$\operatorname{Ex}(f) \cup D$ is an snc divisor.*

This follows from the Main Theorems I and II (or I′ and II′ in the analytic case) of [Hir64]. The result without the assertion (1) is called the Weak Hironaka Theorem, which is all we need in this book. Very short proofs of the Weak Hironaka Theorem for quasi-projective X are given in [AdJ97], [BP96], [Par98]. All these papers reduce the Weak Hironaka Theorem to the torus embedding theory of [KKMSD73]. (They state only that D is an snc divisor, but the proofs work for the full snc statement (3).)

The relative version of resolution is the following:

Theorem 0.3. *Let $f : X \to C$ be a flat morphism of a reduced algebraic variety over $\mathbb{C}$ (or a suitably small neighbourhood of a compact set of a reduced analytic space) to a non-singular curve and $B \subset X$ a divisor. Then there exists a projective birational morphism $g : Y \to X$ from a non-singular Y such that $\mathrm{Ex}(g) + g^*B + (f \circ g)^*(c)$ is an snc divisor for all $c \in C$.*

This follows from the Main Theorem II (or II′) of [Hir64]. It is used only in Chapter 7 with $\dim X = 3$. If C and X are projective, this is a special case of [AK97, Thm. 2.1]. The latter paper also ignores $\mathrm{Ex}(g)$ but the proof again can be modified to yield the full snc statement.

Notation 0.4. In order to avoid possible misunderstanding, here is a list of some of the standard notation we use.

(1) Let X be a normal scheme. A *prime divisor* is an irreducible and reduced subscheme of codimension one. A *divisor* on X is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Z}$. In using this notation we assume that the D_i are distinct. A $\mathbb{Q}$-*divisor* is a formal linear combination $D = \sum d_i D_i$ of prime divisors where $d_i \in \mathbb{Q}$. D is called *effective* if $d_i \geq 0$ for every i. For $\mathbb{Q}$-divisors A, B, we write $A \geq B$ or $B \leq A$ if $A - B$ is effective. (This notation will not be used extensively since it can be easily confused with $A - B$ being nef.) A divisor (or $\mathbb{Q}$-divisor) D is called $\mathbb{Q}$-*Cartier* if mD is Cartier for some $0 \neq m \in \mathbb{Z}$. X is called $\mathbb{Q}$-*factorial* if every $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier. The *support* of $D = \sum d_i D_i$, denoted by $\operatorname{Supp} D$, is the subscheme $\cup_{d_i \neq 0} D_i$.

(2) *Linear equivalence* of two divisors D_1, D_2 is denoted by $D_1 \sim D_2$; *numerical equivalence* of two $\mathbb{Q}$-divisors D_1, D_2 is denoted by $D_1 \equiv D_2$. (We do not define linear equivalence of $\mathbb{Q}$-divisors.) D is said to be *trivial* (resp. *numerically trivial*) if $D \sim 0$ (resp. $D \equiv 0$).

(3) A $\mathbb{Q}$-Cartier divisor D on a proper scheme is called *nef* if $(D \cdot C) \geq 0$ for every irreducible curve $C \subset X$.

(4) A *morphism* of schemes is everywhere defined. It is denoted by a solid arrow $f : X \to Y$. A *map* of schemes is defined on a dense

open set; it is denoted by a dotted arrow $f : X \dashrightarrow Y$. In many books this is called a rational map.

(5) Let $f : X \to Y$ be a morphism and D_1, D_2 two divisors on X. We say that they are *linearly f-equivalent* (denoted by $D_1 \sim_f D_2$) iff there is a Cartier divisor B on Y such that $D_1 \sim D_2 + f^*B$. Two $\mathbb{Q}$-divisors are called *numerically f-equivalent* (denoted by $D_1 \equiv_f D_2$) iff there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on Y such that $D_1 \equiv D_2 + f^*B$. D is said to be *(linearly) f-trivial* (resp. *numerically f-trivial*) if $D \sim_f 0$ (resp. $D \equiv_f 0$).

(6) For a scheme X, $\operatorname{red} X$ denotes the unique reduced subscheme with the same support as X.

(7) For a birational morphism $f : X \to Y$, the *exceptional set* $\operatorname{Ex}(f) \subset X$ is the set of points $\{x \in X\}$ where f is not biregular (that is f^{-1} is not a morphism at $f(x)$). We usually view $\operatorname{Ex}(f)$ as a subscheme with the induced reduced structure.

(8) Let X be a smooth variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X. We say that D is a *simple normal crossing* divisor (abbreviated as *snc*) if each D_i is smooth and they intersect everywhere transversally.

(9) Let X be a scheme. A *resolution* of X is a proper birational morphism $g : Y \to X$ such that Y is smooth.

(10) Let X be a scheme and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor on X. A *log resolution* of (X, D) is a proper birational morphism $g : Y \to X$ such that Y is smooth, $\operatorname{Ex}(g)$ is a divisor and $\operatorname{Ex}(g) \cup g^{-1}(\operatorname{Supp} D)$ is a snc divisor. Log resolutions exist for varieties over a field of characteristic zero by (0.2).

(11) Let $f : X \dashrightarrow Y$ be a map of schemes. Let $Z \subset X$ be a subscheme such that f is defined on a dense open subset $Z^0 \subset Z$. The closure of $f(Z^0)$ is called the *birational transform* of Z. (This is sometimes also called the proper or strict transform.) It is denoted by $f_*(Z)$. If $g : Y \to X$ is birational then we obtain the somewhat unusual looking notation $g_*^{-1}(Z)$. The same notation is used for divisors.

(12) For a real number d, its *round down* is the largest integer $\leq d$. It is denoted by $\lfloor d \rfloor$. The *round up* is the smallest integer $\geq d$. It is denoted by $\lceil d \rceil$. The *fractional part* is $d - \lfloor d \rfloor$ and often denoted by $\{d\}$. If $D = \sum d_i D_i$ is a divisor with real coefficients and the D_i are distinct prime divisors, then we define the *round down* of D as $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, the *round up* of D as $\lceil D \rceil := \sum \lceil d_i \rceil D_i$ and the *fractional part* of D as $\{D\} := \sum \{d_i\} D_i$.

(13) If X is an analytic space, we usually take an arbitrary compact set $Z \subset X$ and work on a suitable small open neighbourhood $U \supset Z$. We may shrink U if it is convenient, without mentioning this explicitly. U is often called the germ of X around Z. If $g : Y \to X$ is a proper morphism of analytic spaces, we usually work over U as above. With these settings, the arguments for algebraic varieties often work and the notation introduced above can be used similarly. Meromorphic maps and bimeromorphic maps are simply called maps and birational maps.

(14) $P := R$ indicates that the new symbol P is defined to be equal to the old expression R.

(15) $\mathbb{Z}_{>0}$ denotes the set of positive integers, and similarly $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers.