First steps *in* Modal Logic

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Survey of propositional logic

1.1 Introduction

Propositional logic is an analysis of the natural language connectives

not
if ... then...
and
or
if ... and only if ...
⋮

as used in a certain restricted context. Thus the analysis is not intended to cover all possible uses of these words in natural language, but only those uses in 'logical arguments' where the meanings of the words are determined in a truth-functional way.

In order to make this context clear the analysis is undertaken via the medium of an abstract, but precisely defined, formal language, the *propositional language.*

The first part of the analysis, the semantics, shows how a truth value can be ascribed to sentences of this language, and then makes precise the notion of the 'logical consequences' of a set of such sentences. This part of the analysis makes use of a standard semantics, i.e. it makes reference to the intended meanings of the connecting symbols of the language (which, of course, are the connectives *not, if...then..., and, ...*).

The second part of the analysis, *propositional calculus*, shows how the notion of 'logical consequence' can be simulated by certain combinatorial manipulations within the language. This is done entirely abstractly without reference to any intended meaning. This simulation can be done in several different ways each making use of a different style of *formal system.* (For classical propositional logic, which is what we are concerned with here, the differences between these styles are more a matter of taste than content.)

The culmination of the analysis is a proof of *completeness*. The chosen formal system is first shown to be *sound* in that anything which is simulated as a logical consequence is one; and then it is shown to be *adequate* in that every logical consequence can be simulated within it.

I assume that, to some extent, you are already familiar with this material. If you are not then you shouldn't be reading this book; there is no point in trying to learn modal logic unless you have a firm grasp of the underlying propositional logic. If you do not have this background I suggest you first acquire it from one (or several) of the many available textbooks covering the subject (some of which are quite good).

In this chapter I will give a brief survey of classical, 2-valued, propositional logic in a form suitable for extension to the modal case. There are many different styles of systems of propositional calculus (Hilbert, Natural, Sequence, ...) all having their good and their bad points. We are not concerned with these pros and cons here; in particular we are not concerned with proof theoretic efficiency (even though this is an important topic which must be addressed eventually). This book is an *introduction* to modal logic, and as such it will present an overview of the basics of the subject rather than the intricacies of the more detailed analysis of certain of its aspects or fields of application.

1.2 The language

So let us begin the refresher course.

The first thing we do is define the abstract, but precisely constructed, *propositional language*. This is built up from certain *primitive symbols* comprising the *variables*, the *connectives*, and the *punctuation symbols*. These are combined in certain ways to produce the *formulas*. The connectives are intended to represent the English language connectives not, if...then..., etc. Since connectives need something to connect, the variables provide a starting point for the process. The punctuation symbols are precisely that; they are used to ensure that the formulas are uniquely readable.

The primitive symbols of the language are:

- The elements $P, Q, R, \ldots$ of a fixed countable set Var of *variables*;
- The *propositional connectives*

$$\top\,,\ \bot\,,\ \neg\,,\ \to\,,\ \wedge\,,\ \vee$$

  of 0, 0, 1, 2, 2, and 2 argument places, respectively;
- The *punctuation symbols* (and).

The formulas of the language are constructed in the usual way.

1.1 DEFINITION. The formulas of the language are obtained recursively using the following clauses.

(atomic) Each variable $P \in Var$ and each constant $\top$ and $\bot$ is a formula.

(propositional) For all formulas θ, ψ, ϕ each of

$$\neg\phi \quad , \quad (\theta \rightarrow \psi) \quad , \quad (\theta \wedge \psi) \quad , \quad (\theta \vee \psi)$$

is a formula.

Let $Form$ be the set of all formulas. ■

The countability of Var is a restriction on the size of the set. If you know what this means then you will recognize where it is used later. If you do not know what it means then, for the purposes of this book, you may regard Var as a given by a list

$$P_0, P_1, P_2, P_3, \cdots, P_r, \cdots$$

However, sometime in the future you should find out what the word means, and how it effects some of the arguments later on.

Note that formulas are defined by a recursion procedure. This means that some facts about formulas can be proved by *structural induction*, i.e. by an induction on the structure of formulas.

For instance, suppose Φ is some set of finite strings of primitive symbols and suppose we know the following.

(0) Φ contains all variables and the two constants $\top$ and $\bot$.

($\neg$) For all formulas θ

$$\theta \in \Phi \quad \Rightarrow \quad \neg\theta \in \Phi.$$

($*$) For all formulas θ and ψ

$$\theta, \psi \in \Phi \quad \Rightarrow \quad (\theta * \psi) \in \Phi$$

(for each binary connective $*$).

We may then conclude that Φ contains all formulas. For suppose not, i.e. suppose there is at least one formula with $\phi \notin \Phi$. Consider an example of such a ϕ containing the least number of symbols. This ϕ can not be a variable or constant, by (0). It must, therefore, have the shape

$$\neg\theta \quad \text{or} \quad (\theta * \psi)$$

for some formulas θ and ψ and connective $*$. But both of these lead to contradictions, by ($\neg$) or ($*$). Thus our original assumption is wrong, hence there is no formula which is not in Φ.

When displaying particular formulas we sometimes omit various brackets and use various other devices to aid readability. However, these displayed strings are not themselves formulas (but just pictures of formulas).

1.3 Two-valued semantics

Let $2 = \{0, 1\}$ and think of 2 as the 'truth object'. We regard 0 as FALSE and 1 as TRUE. Each connective has an associated operation on 2. The operation

$$\neg : 2 \to 2$$

associated with the connective $\neg$ is given by

$$\neg(x) = 1 - x$$

(for each $x \in 2$). Each binary connective $*$ has an associated operation

$$* : 2 \times 2 \to 2$$

given by the following truth table.

		$x * y$		
x	y	$\to$	$\wedge$	$\vee$
0	0	1	0	0
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Notice how this defines the intended meaning of the symbols

$$\neg \qquad \to \qquad \wedge \qquad \vee$$

as

not if ... then and or.

(Note also that we are using the same symbol for the formal connective and its operational counterpart on 2. This should not lead to confusion.)

The basic semantic notion is the construction of the truth value of a formula ϕ. This can not be done in a vacuum, but only within a context where the truth values of the variables are known. The whole process is encapsulated as follows.

1.2 **DEFINITION.** A *valuation* is a map

$$\nu : Var \longrightarrow 2.$$

For each such valuation ν the associated map

$$[\![\cdot]\!]_\nu : Form \longrightarrow 2$$

is defined by recursion on the structure of formulas using the following clauses.

(**Const**) For the constants

$$[\![\top]\!] = 1 \quad , \quad [\![\bot]\!] = 0.$$

(Var) For each variable P

$$[\![P]\!] = \nu(P).$$

($\neg$) For each formula θ

$$[\![\neg\theta]\!] = 1 - [\![\theta]\!].$$

($*$) For all formulas θ, ψ

$$[\![(\theta * \psi)]\!] = [\![\theta]\!] * [\![\psi]\!]$$

(for an arbitrary binary connective $*$). ■

(As in this definition, when using $[\![\cdot]\!]_\nu$, it is usual to drop the distinguishing subscript ν unless this could lead to confusion.)

We say a valuation ν *models* or is a *model* of a formula ϕ, or that ϕ *is true for* ν if

$$[\![\phi]\!] = 1.$$

We can now make precise the notion of 'logical consequence'. Thus, given a set Φ of formulas and a formula ϕ

$$\Phi \models \phi$$

means that ϕ is true for every model of (all members of) Φ. When this holds we say ϕ is a *semantic consequence* of Φ. Formulas ϕ such that

$$\models \phi$$

(i.e. which are true for all valuations) are called *tautologies*.

1.4 The proof theory

The objective of propositional calculus is to give a syntactic description of the semantic consequence relation $\models$ by setting up an appropriate formal system. This can be done in many different ways; here we describe a system that is the most convenient for later generalization to the modal situation. We describe a system in the Hilbert style.

Thus we first set down the set of *logical axioms*. These will be tautologies and typically will contain all formulas of the shapes

$$\begin{array}{ll} (k) & \phi \to (\theta \to \phi) \\ (s) & \theta \to (\psi \to \phi) \,.\!\to\!.\, (\theta \to \psi) \to (\theta \to \phi) \end{array}$$

together with enough axioms to control the other connectives. We also use just one *rule of inference*, modus ponens.

$$\text{(MP)}\ \frac{\theta \qquad \theta \to \phi}{\phi}$$

These are used to generate the proof theoretic consequence relation $\vdash$.

(It is important to notice that we are dealing with the connectives via the use of axioms and not extra rules of inference. In the propositional case there is a relatively easy way to trade off the use of axioms against the use of rules of inference, however in the modal case this is not so easy, so we base our system on just the one rule.)

1.3 DEFINITION. Let Φ be an arbitrary set of formulas.

(a) A *witnessing deduction* from Φ is a sequence

$$\phi_0, \phi_1, \cdots, \phi_n$$

of formulas such that for each formula ϕ_i of the sequence, at least one of the following holds.

(hyp) $\phi_i \in \Phi$.

(ax) ϕ_i is a logical axiom.

(mp) There are formulas ϕ_j, ϕ_k occurring earlier in the sequence (i.e. with $j, k < i$) such that $\phi_k = (\phi_j \to \phi_i)$.

(b) For each formula ϕ, the relation

$$\Phi \vdash \phi$$

holds precisely when there is a witnessing deduction from Φ with ϕ as the last term of this deuction. ■

This relation

$$\Phi \vdash \phi$$

is the simulation of the notion of logical consequence.

Recall that this formal system has the Deduction Property, that is for each set of formulas Φ and pair of formulas θ, ϕ the implication

$$\Phi, \theta \vdash \phi \quad \Rightarrow \quad \Phi \vdash (\theta \to \phi)$$

holds. This is an important property which fails to hold for most modal systems.

1.5 Completeness

It is straight forward to show that the formal system is sound, i.e. that

$$\Phi \vdash \phi \quad \Rightarrow \quad \Phi \models \phi.$$

This is proved by a routine induction on the length of the witnessing formal deduction.

The proof of adequacy (and hence completeness) takes a little longer and can be achieved in several different ways. Here I will sketch a proof which later will form the basis of the corresponding proof for modal systems.

We say a set of formulas Φ is *consistent* if

$$\text{not}[\Phi \vdash \bot].$$

Let $\boldsymbol{CON}$ be the set of all such consistent sets Φ. The formal system is designed to achieve the following properties of $\boldsymbol{CON}$.

(Finite character) For each set of formulas Φ we have $\Phi \in \boldsymbol{CON}$ precisely when $\Psi \in \boldsymbol{CON}$ for each finite $\Psi \subseteq \Phi$.

(Basic consistency) For each variable P we have $\{P, \neg P\} \notin \boldsymbol{CON}$ and, of course, $\{\bot\} \notin \boldsymbol{CON}$.

(Conjunctive preservation) For all appropriate θ, ϕ and Φ with $\Phi \in \boldsymbol{CON}$

$$\begin{aligned}
(\theta \wedge \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\theta, \psi\} \in \boldsymbol{CON} \\
\neg(\theta \vee \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\neg\theta, \neg\psi\} \in \boldsymbol{CON} \\
\neg(\theta \to \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\theta, \neg\psi\} \in \boldsymbol{CON}.
\end{aligned}$$

(Disjunctive preservation) For all appropriate θ, ϕ and Φ with $\Phi \in \boldsymbol{CON}$

$$\begin{aligned}
(\theta \vee \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\theta\} \in \boldsymbol{CON} \quad \text{or} \quad \Phi \cup \{\psi\} \in \boldsymbol{CON} \\
\neg(\theta \wedge \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\neg\theta\} \in \boldsymbol{CON} \quad \text{or} \quad \Phi \cup \{\neg\psi\} \in \boldsymbol{CON} \\
(\theta \to \psi) \in \Phi \quad &\Rightarrow \quad \Phi \cup \{\neg\theta\} \in \boldsymbol{CON} \quad \text{or} \quad \Phi \cup \{\psi\} \in \boldsymbol{CON}.
\end{aligned}$$

(Negation preserving) For all appropriate θ and Φ

$$\neg\neg\theta \in \Phi \in \boldsymbol{CON} \quad \Rightarrow \quad \Phi \cup \{\theta\} \in \boldsymbol{CON}.$$

Now let $\boldsymbol{S}$ be the set of all the maximally consistent sets of formulas, i.e. those $\Phi \in \boldsymbol{CON}$ such that for all sets Ψ

$$\Phi \subseteq \Psi \in \boldsymbol{CON} \quad \Rightarrow \quad \Psi = \Phi.$$

The central pillar which supports the completeness proof is the following existence result.

1.4 LEMMA. (Basic Existence Result) *For each* $\Phi \in \boldsymbol{CON}$ *there is some* $s \in \boldsymbol{CON}$ *with* $\Phi \subseteq s$.

Proof. Let $\{\phi_r \mid r < \omega\}$ be an enumeration of all formulas. Let $(\Delta_r \mid r < \omega)$ be the ascending sequence of sets of formulas defined recursively by

$$\begin{aligned} \Delta_0 &= \Phi \\ \Delta_{r+1} &= \begin{cases} \Delta_r \cup \{\phi_r\} & \text{if this is in } \boldsymbol{CON} \\ \Delta_r & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly $\Delta_r \in \boldsymbol{CON}$ for all $r < \omega$, and hence

$$s = \bigcup\{\Delta_r \mid r < \omega\} \in \boldsymbol{CON}.$$

Finally the construction ensures that $s \in \boldsymbol{S}$. ■

For any $s \in \boldsymbol{S}$ let σ be the valuation given by

$$\sigma(P) = \begin{cases} \text{TRUE} & \text{if } P \in s \\ \text{FALSE} & \text{if } P \notin s \end{cases}$$

(for $P \in Var$). A routine induction now shows that σ is a model of (all the formulas in) s. This makes use of the fact that for $\Phi \in \boldsymbol{S}$ the implications of the preservation properties are equivalences. Thus we have the following.

1.5 THEOREM. *Each* $\Phi \in \boldsymbol{CON}$ *has a model.*

Finally we can achieve the desired completeness result.

1.6 THEOREM. (Completeness) *For each set of formulas* Φ *and formula* ϕ *the equivalence*

$$\Phi \vdash \phi \quad \Leftrightarrow \quad \Phi \models \phi$$

holds.

Proof. The implication ($\Rightarrow$) is soundness, so it suffices to prove ($\Leftarrow$). Thus suppose $\Phi \models \phi$. Then $\Phi \cup \{\neg\phi\}$ has no model and hence Theorem 1.5 gives

$$\Phi \cup \{\neg\phi\} \notin \boldsymbol{CON}.$$

Thus

$$\Phi, \neg\phi \vdash \bot$$

and hence the Deduction Property gives

$$\Phi \vdash (\neg\phi \to \bot)$$

which (with an appropriate axiom) gives

$$\Phi \vdash (\neg\bot \to \phi).$$

Finally, since $\Phi \vdash \neg\bot$, we have $\Phi \vdash \phi$, as required. ■

1.6 Exercises

1.1 Constructing formal derivations can be quite tricky.

(a) Using only the logical axioms (k, s), exhibit witnessing deductions for each of the following.

(i) $\vdash \phi \rightarrow \phi$

(ii) $\vdash (\psi \rightarrow \phi) . \rightarrow . (\theta \rightarrow \psi) \rightarrow (\theta \rightarrow \phi)$

(iii) $\vdash \theta \rightarrow (\psi \rightarrow \phi) . \rightarrow . \psi \rightarrow (\theta \rightarrow \phi)$

(iv) $\vdash (\theta \rightarrow \psi) . \rightarrow . (\psi \rightarrow \phi) \rightarrow (\theta \rightarrow \phi)$

(v) $\vdash (\theta \rightarrow (\theta \rightarrow \psi)) \rightarrow (\theta \rightarrow \psi)$

What are the lengths of these various deductions?

(b) Use the Deduction Property to verify (i - v).

1.2 A set Φ of formulas is said to be *finitely satisfiable* if each finite subset of Φ has a model. Let $\boldsymbol{CON}$ be the set of all finitely satisfiable sets of formulas. Show that $\boldsymbol{CON}$ has the closure properties of Section 1.5, and hence prove the compactness theorem, namely that each finitely satisfiable set of formulas is satisfiable.

1.3 Let P, Q, and R be three finite, pairwise disjoint sets of variables. Let ϕ be a formula built up from $\mathsf{P} \cup \mathsf{Q}$, and let ψ be a formula built up from $\mathsf{Q} \cup \mathsf{R}$. Suppose that

$$\phi \rightarrow \psi$$

is a tautology.

Let Π and Σ be, respectively, the sets of all assignments

$$\mathsf{P} \longrightarrow \mathsf{2} \quad , \quad \mathsf{R} \longrightarrow \mathsf{2}$$

where $\mathsf{2}$ is the truth object. Note that Π and Σ are finite. For each $\pi \in \Pi$ and $\sigma \in \Sigma$ let

$$\phi^\pi \quad , \quad \psi^\sigma$$

be the result of replacing each $P \in \mathsf{P}$ by $\pi(P)$ and each $R \in \mathsf{R}$ by $\sigma(R)$. Let

$$\lambda = \bigvee\{\phi^\pi \mid \pi \in \Pi\} \quad , \quad \rho = \bigwedge\{\psi^\sigma \mid \sigma \in \Sigma\}$$

(so that λ and ρ depend only on Q).

(a) Show that

$$\phi \rightarrow \lambda \quad , \quad \lambda \rightarrow \rho \quad , \quad \rho \rightarrow \psi$$

are tautologies.

(b) Show that for each formula θ built up from $\mathbf{Q}$, if both

$$\phi \to \theta \quad , \quad \theta \to \psi$$

are tautologies, then

$$\lambda \to \theta \quad , \quad \theta \to \rho$$

are also tautologies.

These provide an interpolation result for propositional logic.