

# Gaskinetic Theory

Tamas I. Gombosi

*Professor of Space Science  
Professor of Aerospace Engineering  
The University of Michigan, Ann Arbor*



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# 1

## Introduction

The objective of gaskinetic theory is to explain and predict the macroscopic properties of gases from the properties of their microscopic constituents. Macroscopic properties include the equation of state, specific heats, transport coefficients (such as viscosity, diffusion coefficient, thermal conductivity), etc. The fundamental hypothesis of kinetic theory is that solids, liquids and gases are composed of finite particles (molecules) which are in various states of motion. The term molecule is taken generally to mean a regular arrangement of atoms, or it may refer to a free atom, an electron or an ion.

Statistical mechanics, like kinetic theory, also aims to explain and predict the macroscopic properties of matter in terms of the microscopic particles of which it is composed. The main difference between gaskinetic theory and statistical mechanics is the manner in which particle interactions are treated. In gaskinetic theory the details of the molecular interactions are taken into account. Statistical mechanics avoids these complicated and sometimes poorly known details and replaces them with certain powerful and general statistical ideas.

### 1.1 Brief history of gaskinetic theory

The history of the kinetic theory of gases can be traced back to ancient Greece, where around 400 B.C. Democritus hypothesized that matter is composed of small indivisible particles which he called atoms. The atoms of different materials were assumed to have different sizes and shapes, so that atoms represent the smallest quantity of a substance that retains its properties. Until about two hundred years ago the theory remained just one of several alternative speculations, essentially unsubstantiated by experiments.

The molecular hypothesis entered modern physical science in the eighteenth century. In 1738 Daniel Bernoulli (1700–1782) of Basel explained Boyle's law<sup>1</sup> (the observation that in an isothermal gas the product of the pressure and the vol-

ume remains constant) by assuming that at constant temperature the mean velocity of the gas particles remains constant. In this fundamental work Bernoulli derived a formula equivalent to  $p = \rho u^2/3$ , where  $p$  is the gas pressure,  $\rho$  is the gas mass density and  $u^2$  is the mean square velocity of the gas molecules. The work of Bernoulli can be regarded as the beginning of the modern kinetic theory of gases.

It was not until the general acceptance of the mechanical theory of heat that the kinetic theory of gases was admitted into the mainstream of physical theory. Kinetic theorists also had to explain the specific heats, adiabatic change, and the velocity of sound in gases and provide an alternative theory of matter to replace the Newtonian static, corpuscular theory of gases.

Kinetic theory entered a new phase in 1858 when Rudolf Julius Emanuel Clausius (1822–1888) introduced the concept of mean free path.<sup>2</sup> Clausius' work greatly influenced James Clerk Maxwell (1831–1879) who is considered to be the founder of the modern kinetic theory of gases.

In 1859, Maxwell introduced the concept of the velocity distribution function of molecules for a gas in equilibrium and recognized the equipartition principle of the mean molecular energy for molecules of different mass in a gas mixture.<sup>3</sup> Combining the mean free path and his own concept of velocity distribution function, Maxwell also derived formulae for the transport coefficients of the gas (viscosity, thermal conductivity and diffusion coefficient).<sup>4</sup> Today these papers are regarded as one of the cornerstones of gaskinetic theory.

Another cornerstone is the fundamental integro-differential equation of particle transport derived in 1872 by Ludwig Boltzmann (1844–1906). In his classic paper Boltzmann<sup>5</sup> established the H-theorem, which shows that molecular collisions cause any initial distribution function to relax to a Maxwellian velocity distribution as long as the gas is not influenced by external effects. In the same paper Boltzmann studied non-uniform gases and derived his famous equation for the spatial and temporal evolution of the distribution function which must be satisfied under all conditions.

In 1905 Hendrik Antoon Lorentz (1853–1928), professor of theoretical physics at the University of Leiden, applied Boltzmann's transport theory to the problem of electrical conduction in an electron gas.<sup>6</sup> In fact, this treatment described the electron component of gaseous plasmas, although the significance of the theory was not recognized for a long time.

In 1917 David Enskog (1887–1947) published his Ph.D. dissertation,<sup>7</sup> in which he developed the detailed mathematical theory of non-equilibrium gases. Very similar results were independently published by Sydney Chapman (1888–1970) in a series of papers.<sup>8,9</sup> This highly successful method is the basis of contemporary gaskinetic theory.

In 1949 Harold Grad (1923–1986) developed a new method of solving the Boltzmann equation by expanding the solution into a series of orthogonal polyno-

mials.<sup>10</sup> This method leads to Grad's 13 moment approximation which was widely used from the late 1950s. More recently with the improvement of computer technology the 20 moment approximation is being used for several applications. It can also be shown that the Chapman–Enskog and the Grad methods are mathematically equivalent.

The history of the kinetic theory of gases is full of fascinating discoveries, understandings and misunderstandings. We refer the interested reader to science history books specifically dealing with the emergence of kinetic theory.<sup>11,12</sup>

Today, gaskinetic theory is a widely used tool in various areas of science and engineering. This book concentrates on the kinetic description of transport phenomena with special emphasis on applications in aerospace engineering and space science.

## 1.2 The road to gaskinetic theory in science and engineering

### 1.2.1 Hydrostatics

The practical study of fluid behavior (here the term 'fluid' means both gases and liquids) started a very long time ago with hydrostatics. Hydrostatics is the science of crude engineering approximations to simple fluid phenomena, which was primarily used to solve problems, such as the discharge rate of rivers, hydrostatic pressure on a dam, or wind pressure on a wall, etc. These problems were usually treated with very primitive analysis, which generally goes little beyond very simple principles, such as Bernoulli's law:

$$p = p_0 + \rho g(z_0 - z) \quad (1.1)$$

where  $p$  is pressure,  $\rho$  is the constant mass density,  $g$  is the gravitational acceleration, and  $z$  is altitude (or depth). Problems involving more complicated flows were usually circumvented by the introduction of empirical constants after some form of dimensional analysis.

A typical example for such empirical constants is the discharge coefficient used to describe the discharge rate through an orifice. For instance, consider a reservoir filled with fluid (see Figure 1.1). There is a small orifice with area  $A$  at the bottom of the reservoir. The fluid discharges through the orifice with a flow velocity of  $v=(2gh)^{1/2}$ . The discharge rate of the fluid is  $Q=CAv$ , where  $C$  is the discharge coefficient of the reservoir. This discharge coefficient,  $C$ , has to be determined as an empirical constant from experimental calibration of the orifice. The functional dependence of  $Q$  on  $A$ ,  $h$ , etc., can be formulated by applying either physical intuition or dimensional analysis.

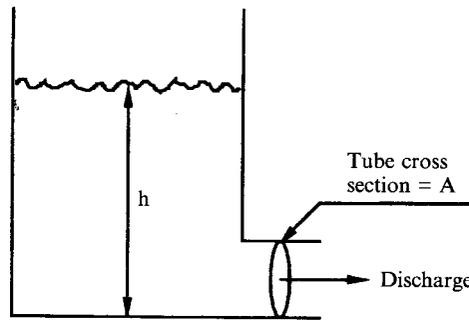


Figure 1.1

In different problems we need different empirical coefficients, i.e., we need drag coefficients to calculate drag, lift coefficients to calculate lift, etc. Things would not be too bad if the empirical coefficients were really constant over a wide range of experimental conditions and for various fluids. This is not the case because in using a coefficient we are taking gross averages of quantities characterizing physical phenomena. The physical process is usually too complicated to be represented by such a simple parameter.

### 1.2.2 Hydrodynamics (*fluid mechanics*)

Hydrodynamics represents a more sophisticated approach to the description of fluids than hydrostatics. The main difference is the much more rigorous and refined use of physical and mathematical principles. The basic idea of hydrodynamics is to take a small fluid element in the flow field and consider the forces acting on it and to calculate the resulting velocities and the changes of the element as a function of space and time. The basic relations are appropriately formulated conservation laws of mechanics, such as the conservation of mass, momentum and energy.

The internal state of the fluid element is determined by laws of thermodynamics and the equation of state. The ensuing relations combined with initial and boundary conditions define the mathematical problem which is usually a set of differential equations. The solutions 'almost' describe the flow field: almost, because some empirical coefficients are still needed. In hydrodynamics one still needs a few basic empirical coefficients or relations describing some internal properties of the fluid. These empirical coefficients include: (i) the viscosity coefficient, which relates the viscous stress to the rate of strain of the fluid element (velocity gradient); (ii) the heat conduction coefficient, which relates the heat flow to the temperature gradient; (iii) the diffusion coefficient, which relates the mass transfer rate to the gradients of concentration, temperature and pressure.

These fundamental coefficients and the equation of state depend on the internal structure of the fluid: their values can be determined through experimental calibrations of different fluids and have a wide range of validity.

Compared with hydrostatics, fluid dynamics is a big step forward. Instead of empirical coefficients for each individual flow problem we need only a few basic coefficients which apply to a large family of problems. Hydrodynamics also provides much more information about the flow field. This is an advantage only if we can solve the hydrodynamic equations for a given flow problem.

Hydrodynamics is not a new science. It had been in existence for a long time before physicists and engineers discovered it. In the 18th century it was a favorite field for mathematicians since it involved interesting aspects of the theory of differential equations. A number of fundamental differential equations were solved for flow problems.

In the 19th century, hydrodynamics played with paradoxes, such as the interpretation of drag force on a body in a stream of perfect fluid. Hydrodynamics was not considered physically realistic until Ludwig Prandtl<sup>13</sup> worked out the boundary layer theory and resolved many paradoxes. His contribution transformed hydrodynamics from a subject of pure academic interest to a discipline of enormous physical and engineering significance.

Compressible fluid dynamics deals with gases and concentrates on the description of subsonic, transonic, supersonic and hypersonic flows. This is generally called gas dynamics. As more and more problems are created by science and technology, new physical phenomena and problems are encountered. These new challenges include the increasing speed and energy range in flows, flows at very low density (such as high altitude flight, spacecraft re-entry, interplanetary flight, space science), regions of very large gradients (shock waves), flows with chemical reactions, flows of ionized species (plasma physics), and non-equilibrium flows.

In these cases we repeatedly encounter the problem that the classical gas transport coefficients (viscosity, heat conduction) lose their meaning, because the internal state of the gas becomes too complicated and the coefficients cease to be constant. In addressing these new problems, it is not sufficient to consider the continuum model generally used by gas dynamics. We must build the flow theory from a more basic point of view. This new approach is the molecular approximation, which leads to gaskinetic theory.

### 1.2.3 Gaskinetic theory

The next step in the description of the fluid phenomena is kinetic theory which is capable of deriving transport coefficients from the fundamental properties of the gas molecules. Gaskinetic theory is based on the concept that a 'gas' is a collection

of very many particles which are acted upon by their surroundings and by mutual encounters. The molecules are too small to be seen, but they obey the classical laws of matter (possibly modified to describe some quantum effects, too).

The classical kinetic theory of gases emerged from a combination of mechanics and statistics. The motions of the molecules are described by probability and not by their individual paths. This approach is the kinetic theory of gases, which is based on first principles and has led to an improved description of the pressure, temperature and equation of state, as well as the description of transport properties (viscosity, thermal conductivity, diffusion coefficients).

This book is an introduction to the theory of kinetic theory of gases, that is the molecular theory of compressible fluids. The foundations of molecular theory will be used to interpret those flow quantities which cannot be determined from the continuum theory of fluids.

### 1.3 Basic assumptions of gaskinetic theory

The kinetic theory of gases is based on three fundamental approximations. These are the molecular hypothesis, the assumption that classical conservation laws can be applied, and the application of statistical methods.

#### 1.3.1 Molecular hypothesis

The fundamental assumption that all matter is composed of small elementary building blocks can be traced to ancient Greek philosophers. The molecular hypothesis assumes that (i) matter is composed of small discrete units (called molecules), (ii) the molecules are the smallest quantity of a substance that retains its chemical properties, (iii) all molecules of a given substance are alike, and (iv) the states of matter (solids, liquid, gas) differ essentially in the arrangement and state of motion of the molecules.

Gases differ from liquids and solids by completely filling any container in which they are placed, by being extraordinarily compressible, by rapidly diffusing into one another, and by having very low densities. These facts suggest that the molecules in a gas are widely separated from one another (meaning that the average distance between molecules is much greater than the size of the molecules) and that the molecules move about throughout the entire space occupied by the gas.

In a gas, the molecules move about freely and most of the time are separated by distances that are large compared to their dimensions. They move in straight lines until two (or more) happen to come so close together that they act strongly upon each other and something like a collision occurs, after which they separate and

move off in new directions, generally with different velocities.

This book will primarily apply kinetic theory to ideal (perfect) gases. We define a perfect gas using the following assumptions:

- (i) molecules are point-like with no internal structure or internal degrees of freedom (perfect gases are monatomic);
- (ii) molecules exert forces on each other only when their centers are within a distance which is very small compared to their average separation (the region in which the particles interact is called the sphere of influence);
- (iii) outside each other's sphere of influence molecules obey the laws of classical mechanics (their motion is described by a non-relativistic equation of motion).

In perfect gases heat energy is interpreted as random translational energy of the molecules. However, molecules which possess internal structure may store energy in their internal degrees of freedom (kinetic energy of rotation and vibration or potential energy of vibration). Although classical kinetic theory makes specific predictions of how the kinetic energy is partitioned among all degrees of freedom, the results do not agree with observations except in the cases of the simplest molecules. This failure clearly shows the limits of classical gaskinetic theory.

Friction is entirely absent in molecular interactions. In kinetic theory the conversion of mechanical work into heat by friction is interpreted as a conversion of bulk translational energy into random molecular motion. After the body has been brought to rest by friction, there may be as much motion as before, only the formerly organized motion now becomes part of the completely disorganized heat motion.

### 1.3.2 Classical conservation laws

It should be repeatedly emphasized that relativistic corrections, quantum mechanical and non-perfect gas effects are usually not considered in the classical kinetic theory of gases. The notable exceptions are the calculation of specific heats and collision cross sections, where quantum mechanical effects usually must be taken into consideration.

The internal structure of atoms and molecules enters into kinetic theory in two essentially different ways. Quantum mechanical effects are relatively unimportant in those parts of gaskinetic theory which treat only the translational motion of molecules (such as viscosity and diffusion). In these problems the internal molecular structure (which is governed by quantum mechanics) matters only so far as it determines the exact nature of the intermolecular forces. On the other hand it should be recognized that the intermolecular force laws in gaskinetic theory are taken as semi-empirical expressions. Quantum mechanical theory of atomic and molecular structure provides a method for obtaining these semi-empirical force laws.

Quantum mechanical effects must be taken into account in any gaskinetic problem in which the energy stored in internal degrees of freedom is considered. The molecular rotation and vibration and the motion of electrons must all be treated by methods of quantum theory, and the classical methods are correct only for sufficiently high temperatures.

### 1.3.3 Statistical nature of the theory

The kinetic theory of gases is a statistical theory: in addition to dynamical methods it also uses a statistical approach. The dynamical description of the motion of a single particle is given by the non-relativistic equation of motion, which is a second order differential equation with two integration constants for each translational degree of freedom. If the values of these integration constants are specified (by the initial conditions) and all the forces acting on the particle are known at all spatial locations and at all times, its entire future dynamical behavior is fully described by the equation of motion.

Regarding a gas as a system of  $N$  particles (each with three translational degrees of freedom), there would be altogether  $N$  vector equations of motion with  $6N$  integration constants. The problem is that for typical gases  $N$  is a prohibitively large number. There are a very large number of molecules in every 'macroscopically infinitesimal' sample.

In order to illustrate this point Figure 1.2 shows the gas concentration as a function of altitude<sup>14</sup> in the atmosphere of Earth between the surface and 1000 km. It can be seen that there are about  $10^{25}$  molecules in a cubic meter at the surface. To solve  $\geq 10^{25}$  equations is not easy. Besides, it is not needed. Complete knowledge of the history of each molecule in the gas is not what we are seeking. We need to know only so much about the molecular motions as is necessary to under-

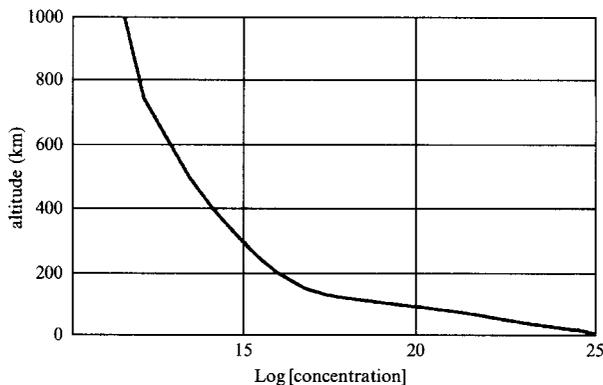


Figure 1.2

stand and predict the observable, macroscopic properties of the gas. The measurement of these quantities involves taking a time average over the instantaneous states of motion of the individual particles in ‘macroscopically infinitesimal’ volumes.

These average quantities might be the density, bulk velocity, pressure, etc. Gaskinetic theory involves the description of a very large number of individual molecules by statistical methods. This description is meaningful, because we have a large number of molecules in any macroscopically infinitesimal volume.

### 1.4 Notations

Throughout this book vectorial quantities either are denoted by coordinate subscripts (such as  $x_i$ , where  $i=1, 2, 3$ ) or are printed in boldface type ( $\mathbf{x}$ ). Throughout the book coordinate indices are denoted by  $i, j, k, l, m$  and  $n$ . When using Cartesian coordinates this notation means  $x_1=x$ ,  $x_2=y$ , and  $x_3=z$ . In expressions written in component notation, summation must be applied with respect to coordinate indices which occur twice in one term. This means that the dot product of vectors  $\mathbf{x}$  and  $\mathbf{v}$  can be denoted either as  $\mathbf{x} \cdot \mathbf{v}$  or as  $x_i v_i$ .

The differential operator vector,  $\nabla$ , is sometimes also written in component notation,  $\nabla=(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ . The velocity space del operator is denoted by  $\nabla_v$ , and it can be written in component notation as  $\nabla_v=(\partial/\partial v_1, \partial/\partial v_2, \partial/\partial v_3)$ .

The subscripts  $s$  and  $t$  refer to different particle species. Summation with respect to these subscripts is always indicated explicitly (i.e.  $\sum_t f_t$ ).

The Kronecker symbol,  $\delta_{ij}$ , is equal to +1 when the two indices are equal, and it is equal to zero when the two indices are different. For instance,  $\delta_{11}=1$  and  $\delta_{12}=0$ .

The permutation tensor,  $\epsilon_{ijk}$ , is equal to zero when at least two indices are equal, it has the value of +1 when the three subscripts are all different and form an even permutation of 1,2 and 3, and finally it takes the value of -1 when the three subscripts are all different and form an odd permutation. For instance,  $\epsilon_{312}=1$ ,  $\epsilon_{113}=0$ , and  $\epsilon_{213}=-1$ . The permutation tensor is necessary to express vector products in component notation. For instance the  $i$ -th component of the vector product of vectors  $\mathbf{x}$  and  $\mathbf{v}$  can be denoted either as  $(\mathbf{x} \times \mathbf{v})_i$  or as  $\epsilon_{ijk} x_j v_k$ .

Using our index notation the trace of tensor  $A$  can be written as  $A_{ii}$ . The  $i, j$  element of the unit tensor is obviously  $I_{ij}=\delta_{ij}$ . The trace of the unit tensor is  $\delta_{ii}=3$ . Tensors with two or more indices can be contracted to obtain lower order tensors. For instance, by contracting the two indices of a regular matrix we get a scalar quantity,  $a=A_{ii}$ . By contracting two indices of a three index tensor we get a vector,  $q_i=Q_{ijj}$ .

## 1.5 Solid angles and curvilinear coordinates

### 1.5.1 Spherical coordinates and solid angles

In gaskinetic theory one usually has to deal with problems involving seven independent variables: time, three components of the position vector and three components of the velocity vector. Every symmetry of the problem can help us to reduce the number of independent variables and thus make the problem somewhat more tractable. The most commonly used symmetry assumptions in gaskinetic theory are those of spherical and cylindrical geometries, therefore the use of spherical or cylindrical coordinates can result in significant mathematical simplifications. In this section we briefly outline the basic concepts of solid angles, spherical and cylindrical coordinate systems.

The spherical coordinates  $(r, \theta, \phi)$  of a point, P, denote the distance of the point from the origin of the coordinate system, the polar angle measured between the  $+z$  axis and the radius vector, and the azimuth angle measured between the  $+x$  axis and the projection of the radius vector onto the  $(x, y)$  plane. The spherical coordinates,  $r$ ,  $\theta$ , and  $\phi$ , vary from 0 to  $\infty$ , from 0 to  $\pi$ , and from 0 to  $2\pi$ , respectively (the range of the angle,  $\theta$ , is constrained to avoid redundancy). This coordinate system is shown in Figure 1.3. Figure 1.3 also shows the new orthogonal unit vectors,  $\mathbf{e}_r$ ,  $\mathbf{e}_\theta$ , and  $\mathbf{e}_\phi$ . The relation between the spherical and Cartesian coordinates can be expressed as

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \tag{1.2}$$

If  $r$  and  $\theta$  are fixed and  $\phi$  is varied, P moves along a parallel of latitude, which maps out a circle of radius  $r \sin \theta$ . If the azimuth angle changes by  $d\phi$ , the arc length

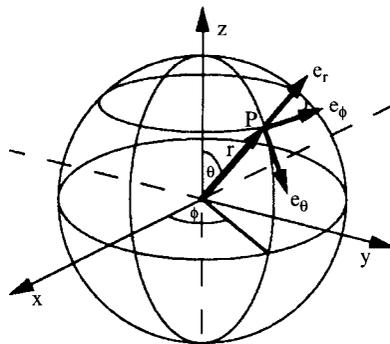


Figure 1.3

along this circle changes by  $r \sin \theta d\phi$ . If  $r$  and  $\phi$  are fixed and  $\theta$  is varied,  $P$  moves along a meridian. If the polar angle changes by  $d\theta$ , the arc length along the meridian changes by  $r d\theta$ . If  $r$  is fixed and  $\theta$  and  $\phi$  are varied, the point,  $P$ , moves on the surface of a sphere of radius  $r$ . If the azimuth angle changes by  $d\phi$  and the polar angle changes by  $d\theta$ , the corresponding area element on the surface of the sphere is

$$dS = r^2 \sin \theta d\theta d\phi \quad (1.3)$$

If the radius vector is also allowed to change, i.e. the radius changes by  $dr$ , the azimuth angle changes by  $d\phi$  and the polar angle changes by  $d\theta$ , the corresponding volume element can be expressed as

$$dV = d^3r = r^2 \sin \theta d\theta d\phi dr \quad (1.4)$$

Here and throughout this book three dimensional volume elements will be denoted by  $d^3r$ . It is obvious that in Cartesian coordinates  $d^3r = dx dy dz$ .

The solid angle is a generalization to three dimensions of the radian definition of angle in a plane. If  $s$  is the length of arc along a circle of radius  $r$ , then the central angle,  $\alpha$ , is defined as  $\alpha = s/r$ . In three dimensions the circle is replaced by a sphere of radius  $r$ , and the arc is replaced by a surface area on the sphere,  $S$ , bounded by an arbitrary closed curve  $C$ . The definition of the solid angle covered by the surface area,  $S$ , as seen from the center of the sphere, is  $\Omega = S/r^2$ . Since the largest value of  $S$  is  $4\pi r^2$ , one can immediately see that  $\Omega$  can vary from 0 to a maximum value of  $4\pi$ . In analogy to plane angles the solid angle is measured in units of steradian.

Using the above definition of solid angle one can obtain the infinitesimal solid angle element occupied by the surface element  $dS = r^2 \sin \theta d\theta d\phi$ :

$$d\Omega = \frac{dS}{r^2} = \sin \theta d\theta d\phi \quad (1.5)$$

Sometimes it is more convenient to replace the polar angle,  $\theta$ , by its cosine,  $\mu = \cos \theta$ . In this case the spherical coordinates are  $r$ ,  $\mu$  and  $\phi$ , with  $\mu$  changing from 1 ( $\theta=0$ ) to  $\mu=-1$  ( $\theta=\pi$ ). In this case the infinitesimal surface element can be expressed as  $dS = -r^2 d\mu d\phi$ . The negative sign can be eliminated by interchanging the integration boundaries: in this case the integral over  $\mu$  goes from  $-1$  to  $+1$ . When one integrates over the entire solid angle the integrals go from  $\mu=-1$  to  $\mu=+1$  and from  $\phi=0$  to  $\phi=2\pi$ .

Some fundamental differential operators in spherical coordinates are given in the Appendix.

### 1.5.2 Cylindrical coordinates

In some cases problems show rotational symmetry about a specific axis. In such cases the use of cylindrical coordinates might offer great mathematical simplifications.

The cylindrical coordinates  $(\rho, \phi, z)$  of a point P denote the projection of the radius vector to the  $(x, y)$  plane, the azimuth angle in the  $(x, y)$  plane measured counterclockwise from the  $x$  axis, and the  $z$  coordinate, respectively. This coordinate system is shown in Figure 1.4 together with the curvilinear orthogonal unit vectors,  $\mathbf{e}_\rho$ ,  $\mathbf{e}_\phi$ , and  $\mathbf{e}_z$ . It can be seen from Figure 1.4 that the cylindrical coordinates,  $\rho$ ,  $\phi$ , and  $z$  vary from 0 to  $\infty$ , from 0 to  $2\pi$ , and from  $-\infty$  to  $+\infty$ , respectively. Using elementary geometric considerations one can readily express the relation between the cylindrical and Cartesian coordinates in the following form:

$$\begin{aligned} x &= \rho \cos \phi \\ y &= \rho \sin \phi \\ z &= z \end{aligned} \tag{1.6}$$

The three dimensional infinitesimal volume element can be expressed in cylindrical coordinates as

$$d^3r = \rho d\phi dr dz \tag{1.7}$$

Some fundamental differential operators in cylindrical coordinates are given in the Appendix.

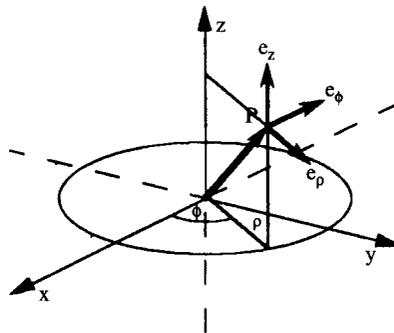


Figure 1.4

### 1.6 Problems

1.1 Using index notation show that  $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ , where the vector,  $\mathbf{B}$ , is a function of location,  $\mathbf{r}$ .

1.2 In a vacuum the vector form of Maxwell's equations is the following:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \times \mathbf{B} &= \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} \end{aligned}$$

where  $\mathbf{E}$  and  $\mathbf{B}$  are the electric and magnetic field vectors,  $\rho$  is the electric charge density,  $\mathbf{j}$  is the electric current density, and  $\mu_0$  and  $\epsilon_0$  are the magnetic permeability and electric permittivity of free space, respectively. Write these equations using index notation.

1.3 Comet Halley has a total surface area of 500 km<sup>2</sup>. According to the images made by the Giotto spacecraft, cometary activity was concentrated in a 50 km<sup>2</sup> area on the sunlit side. Approximate the nucleus by a sphere and calculate the solid angle covered by the active area.

### 1.7 References

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