# Contents

*Foreword*  
page xi

*Acknowledgements*  
xviii

*General Introduction: Simplicity, Economy, Elegance*  
Chris Pritchard  
1

## Part I: The Nature of Geometry  
11

1.1 What is geometry?  *G H Hardy* (1925)  
13

24

## Desert Island Theorems  
31

**Group A: Greek Geometry**  

A1  Pythagoras’ Theorem  *Chris Denley*  
33

A2  Angle at centre of a circle is twice angle at circumference  *Charlie Stripp*  
35

A3  Archimedes’ theorem on the area of a parabolic segment  
*Tom Apostol*  
37

A4  An isoperimetric theorem  *John Hersee*  
90

A5  Ptolemy’s Theorem  *Tony Crilly and Colin Fletcher*  
42

## Part II: The History of Geometry  
51

2.1 Introductory Essay: A concise and selective history of geometry  
from Ur to Erlangen  *Chris Pritchard*  
53

2.2 Greek geometry with special reference to infinitesimals  
*Sir T. L. Heath* (1923)  
73

2.3 A straight line is the shortest distance between two points  
90

2.4 On geometrical constructions by means of the compass  
*E W Hobson* (1913)  
92

2.5 What is a square root? A study of geometrical representation in  
different mathematical traditions  *George Gheverghese Joseph*  
(1997)  
100
2.6 An old Chinese way of finding the volume of a sphere  
*T Kiang* (1972) 115

2.7 Mathematics and Islamic Art  *Lesley Jones* (1989) 119

2.8 Jamshīd al-Kāshī, calculating genius  *Glen Van Brummelen* (1998) 130

2.9 Geometry and Girard Desargues  *B A Swinden* (1950) 136

2.10 Henri Brocard and the geometry of the triangle  
*Laura Guggenbuhl* (1953) 146

2.11 The development of geometrical methods  *Gaston Darboux* (1904) 150

**Desert Island Theorems**

**Group B: Elementary Euclidean Geometry** 173

B1 Varignon’s Theorem  *Chris Pritchard* 175

B2 Varignon’s big sister?  *Celia Hoyles* 177

B3 Mid-Edges Theorem  *Toni Beardon* 179

B4 Van Schooten’s Theorem  *Doug French* 184

B5 Ceva’s Theorem  *Elmer Rees* 187

B6 Descartes Circle Theorem  *H S M Coxeter* 189

B7 Three Squares Theorem  *Bill Richardson* 193

B8 Morley’s Triangle Theorem  *David Burghes* 195

**Part III: Pythagoras’ Theorem** 199

3.1 Introductory Essay: Pythagoras’ Theorem, A Measure of Gold  
*Janet Jagger* 201

3.2 Pythagoras  *Walter Rouse Ball* (1915) 211

3.3 Perigal’s dissection for the Theorem of Pythagoras  
*A W Siddons* (1932) 222

3.4 Demonstration of Pythagoras’ Theorem in three moves  
*Roger Baker* (1997) 224

3.5 Pythagoras’ Theorem  *Jack Oliver* (1997) 226

3.6 A neglected Pythagorean-like formula  *Larry Hoehn* (2000) 228

3.7 Pythagoras extended: a geometric approach to the cosine rule  

3.8 Pythagoras in higher dimensions, I  *Lewis Hull* (1978) 237

3.9 Pythagoras in higher dimensions, II  *Hazel Perfect* (1978) 239

3.10 Pythagoras inside out  *Larry Hoehn* (1996) 240

3.11 Geometry and the cosine rule  *Colin Dixon* (1997) 242

3.12 Bride’s chair revisited  *Roger Webster* (1994) 246


**Desert Island Theorems**

**Group C: Advanced Euclidean Geometry** 251

C1 Desargues’ Theorem  *Douglas Quadling* 253
Contents

C2 Pascal’s Hexagram Theorem  Martyn Cundy  257
C3 Nine-point Circle  Adam McBride  261
C4 Napoleon’s Theorem and Doug-all’s Theorem  Douglas Hofstadter  265
C5 Miquel’s Six Circle Theorem  Aad Goddijn  272
C6 Eyeball Theorems  Antonio Gutierrez  274

Part IV: The Golden Ratio  281

4.1 Introduction  Ron Knott  283
4.2 Regular pentagons and the Fibonacci Sequence  Doug French (1989)  286
4.3 Equilateral triangles and the golden ratio  J F Rigby (1988)  292
4.4 Regular pentagon construction  David Pagni (2000)  296
4.5 Discovering the golden section  Neville Reed (1997)  299
4.6 Making a golden rectangle by paper folding  George Markowsky (1991)  301
4.7 The golden section in mountain photography  David Chappell and Christine Straker (1990)  304
4.8 Another peek at the golden section  Paul Glaister (1990)  306
4.9 A note on the golden ratio  A D Rawlins (1995)  308
4.10 Balancing and golden rectangles  Nick Lord (1995)  310
4.11 Golden earrings  Paul Glaister (1996)  312
4.12 The pyramids, the golden section and $2\pi$  Tony Collyer and Alex Pathan (2000)  314
4.13 A supergolden rectangle  Tony Crilly (1994)  320

Desert Island Theorems

Group D: Non-Euclidean Geometry & Topology  327

D1 Four-and-a-half Colour Theorem  Derek Holton  329
D2 Euler-Descartes Theorem  Tony Gardiner  333
D3 Euler-Poincaré Theorem  Carlo Séquin  339
D4 Two Right Tromino theorems  Solomon Golomb  343
D5 Sum of the angles of a spherical triangle  Christopher Zeeman  346

Part V: Recreational Geometry  349

5.1 Introduction  Brian Bolt  351
5.2 The cube dissected into three yángmā  James Brunton (1973)  357
5.3 Folded polyhedra  Cecily Nevill (1996)  359
5.4 The use of the pentagram in constructing the net for a regular dodecahedron  E M Bishop (1962)  362
5.5 Paper patterns: solid shapes from metric paper  William Gibbs (1990)  363
5.6 Replicating figures in the plane  Solomon Golomb (1964)  367
5.7 The sphinx task centre problem  Andy Martin (2000)  371
5.8 Ezt Rakd Ki: A Hungarian tangram  Jean Melrose (1998) 378
5.9 Dissecting a dodecagon Doug French (1998) 383
5.10 A dissection puzzle Jon Millington (1993) 389
5.11 Two squares from one Brian Bolt (2001) 393
5.12 Half-squares, tessellations and quilting Tony Orton (1994) 396
5.13 From tessellations to fractals Tony Orton (1991) 401
5.14 Paper patterns with circles William Gibbs (1990) 406
5.15 Tessellations with pentagons [with related correspondence] J A Dunn (1971) 421
5.16 Universal games Helen Morris (1997) 428

Desert Island Theorems

Group E: Geometrical Physics 443
E1 Euler’s Identity Keith Devlin 445
E2 Clifford Parallels Michael Atiyah 448
E3 Tait Conjectures Ruth Lawrence 450
E4 Kelvin’s Circulation Theorem Keith Moffatt 455
E5 Noether’s Theorem Leon Lederman and Chris Hill 456
E6 Kepler’s Packing Theorem Simon Singh 459

Part VI: The Teaching of Geometry 461
6.1 Introductory Essay: A century of school geometry teaching Michael Price 463
6.2 The teaching of Euclid Bertrand Russell (1902) 486
6.3 The Board of Education circular on the teaching of geometry Charles Godfrey (1910) 489
6.4 The teaching of geometry in schools C V Durell (1925) 496
6.5 Fifty years of change A W Siddons (1956) 500
6.6 Milestone or millstone? Geoffrey Howson (1973) 505
6.7 The place of geometry in a mathematical education J V Armitage (1973) 515

Appendices 527
Appendix I: Report of the M. A. Committee on Geometry (1902) 529
Appendix II: Euclidean Propositions 537
Index 542
As children we build sandcastles and snowmen, construct buildings with LEGO and play computer games that create the impression of rapid movement through three-dimensional space. In later life, we hang wallpaper, negotiate narrow spaces in our cars and tease furniture through doorways. A surprising number of us use specially-honed spatial skills to earn a living, such as barbers and sculptors, interior designers and footballers, bus-drivers and architects, supermarket shelf-stackers, couturiers and civil engineers. Occasionally a Pelé, an Yves Saint Laurent or a Barbara Hepworth comes along. But regardless of the extent to which our spatial talents are developed, from the cradle to the grave, we are all geometers.

This book celebrates the best of geometry in all its simplicity, economy and elegance. Such a wonderful tray of attributes might have formed a ‘splendid title’ for the book as a whole, in the view of H. S. M. Coxeter, but instead it serves as a theme for the Desert Island Theorems and as the title of this general introduction. The source is a lovely anecdote about Peter Frazer, related by H. E. Huntley in his book *The Divine Proportion* [1, p. 5]:

Peter Frazer . . . a lovable man and a brilliant teacher, was discussing cross ratios with a mathematics set. Swiftly, he chalked on the blackboard a fan of four straight lines, crossed them with a transversal and wrote a short equation; he stepped down from the dais and surveyed the figure . . . Striding rapidly up and down between the class and the blackboard, waving his arms about excitedly, with his tattered gown, green with age, billowing out behind him, he spoke in staccato phrases: ‘Och, a truly beautiful theorem! Beautiful! . . . Beautiful! . . . Look at it! Look at it! What simplicity! What economy! Just four lines and one transversal . . . What elegance!’

It appears that Frazer saw simplicity, economy and elegance as complementary essences of beauty in geometry. If I were to add a fourth attribute, it would be ‘surprise’. To elaborate on this theme consider two results from elementary Euclidean geometry. Firstly, in the figure, $ABCD$ is a square and $E, F, G, H$ are
the midpoints of $AB$, $BC$, $CD$, $DA$ respectively. Then:

$$\text{Area } IJKL = \frac{1}{5} \text{ Area } ABCD.$$ 

A simple dissection proof consists of rearranging the figure into a pentomino in the form of a cross by rotating $\triangle EJB$ through 180 degrees about $E$, and rotating the other three small triangles in like manner.

Secondly, take two maps of the same area, one a pocket map and the other a comprehensive map on a larger scale. (Alternatively, take two differently-sized prints of the same photograph.) Place the pocket map on the detailed map at any angle. Then there is one and only one point on the pocket map lying immediately on top of the point corresponding to it on the map beneath. This can be demonstrated using simple transformation geometry. First rotate the pocket map $ABCD$ about the common point, $E$, to give $MNPQ$ with the same orientation as the larger map.
General Introduction

The common point then acts as the centre of a dilatation (or enlargement) of the pocket map. Now let us reverse the argument. Given the two maps with the smaller cast carelessly onto the larger, how would we go about finding the common point? (A solution using cyclic quadrilaterals is shown at the end of the introduction.)

The Mathematical Association came into existence in 1871 at a time when geometry teaching was in something of a tumult. The school and university geometry curricula and Euclid’s *Elements* were as two peas in a pod. So completely was the curriculum determined by the standard text that Sylvester sarcastically referred to the *Elements* as ‘one of the advanced outposts of the British Constitution’. Calls for reform in the universities met with the antipathy of De Morgan, Cayley and Kelland and with positions entrenched, it fell to schoolteachers to advance the case for ‘loosening the shackles’ of Euclid [2, ch. 2].

The sort of issues that challenged teachers at the time, concerned the choice and order of the Euclidean propositions to be taught, to whom and at what age? The role of riders in geometry education was also considered. Geometry continued to be viewed as the ideal vehicle for developing an understanding of formal proof. It is only in recent decades, with the advent of modern mathematics and its emphasis on transformation geometry, that the status of formal proof has fallen away and its ‘natural habitat’ relocated to algebra. We stand on the threshold of a possible reversal of this trend, though as ever it will be difficult to reach a consensus.

In its first thirty years, the movement to improve geometry teaching had only limited success. But by the turn of the century there was a more general acknowledgement of the problems by both educationalists and administrators and the time was ripe for significant change. A growing understanding of the need to broaden the scope and methods of geometry in schools saw the first use of measuring instruments such as protractors and the adoption of four-figure tables.

To digress a little, at high school in the late 1960s, my classmates and I made use of such four-figure tables, compiled according to the frontispiece by ‘the late C. Godfrey and the late A. W. Siddons’. We had the great good fortune to be taught geometry by Wil Williams, a teacher of unusual clarity and not a little humour – “they died calculating the entries”, was his quip. At the time, little did he or we know of the enormous roles Godfrey and Siddons played in The Mathematical Association. Twenty-three of its early members formed the first Teaching Committee in 1902 and Godfrey and Siddons were prominent among them. This book celebrates the centenary of Teaching Committee, a body which in its early years helped to subjugate the geometry text to the requirements of the designed curriculum and which has continued ever since to provide advice and resources for mathematics teachers and to seek to influence national policies on mathematics education.
At the Annual Meeting of the National Council of Teachers of Mathematics in Florida in April 2001, I sought the views of one of its senior figures on The Mathematical Association. It was no great surprise that the first thing he said in reply was that the association has very good journals. And indeed, the journals attract contributions from all parts of the world. Since it is from two of these journals, the Mathematical Gazette and Mathematics in School, that a goodly proportion of the material for this book is taken, a little more information about them is in order.

The Mathematical Gazette, the premier journal of The Mathematical Association, was established in April 1894 under the editorship of Edward Mann Langley. The opening editorial made it plain that it would enable mathematics teachers to share successful approaches to their art, other than those to be found in the texts of the day [2, p. 40]. In this role it has met with fluctuating success during its 108-year history, whilst maintaining greater consistency as a minor mathematical serial. It was Langley, incidentally, who as a teacher at Bedford Modern School converted E. T. Bell to mathematics and quite likely helped shape that accomplished number theorist and somewhat unreliable mathematical biographer [3, ch. 2]. Bell duly contributed an article to the Mathematical Gazette’s 250th number in July 1938 and, with its reputation soaring both in Britain and internationally, the 500th number was published in July 2000. The launch of Mathematics in School in November 1971 provided members with a journal designed to support the teaching of mathematics to younger children. Its style remains to this day rather more informal than that of the Mathematical Gazette, with a larger proportion of articles focusing on classroom practice, often with a hands-on flavour. Elementary geometry and its teaching have formed the subject of several thousand articles which have appeared in these two journals over the last century; many of the best get a second airing in this volume.

Following an initial trawl some twenty geometrical themes were identified for possible inclusion in this book. Each perhaps merited a separate part but six were finally selected for inclusion. The first two parts, on the nature and history of geometry, and the last part, on its teaching, are somewhat weightier than the other three on Pythagoras’ Theorem, the golden section and recreational geometry. They tend to set the scene, to help the reader to understand the contexts within which geometry and its teaching developed. A number of articles in these sections refer to Euclidean definitions and theorems simply by taking the book number and the proposition number from the Elements. Thus, Pythagoras’ Theorem is Euclid I, 47 because it is the forty-seventh proposition of the first book of the Elements. Note that Euclid and his book are synonymous. Since a large proportion of a modern audience will be unfamiliar with these tags, each of the propositions referred to in this way has been stated in full in the second appendix.
Two of the most illustrious mathematicians of this century, G. H. Hardy and Michael Atiyah offer views on what constitutes geometry. Hardy confesses from the outset that ‘I do not claim to know any geometry, but I do claim to understand quite clearly what geometry is’ and comes to a pessimistic conclusion about ever reaching a consensus on the geometry curriculum. Atiyah takes an historical approach to tease out the nature of the subject.

Following a general overview of the history of geometry, the history articles open with an essay on Greek mathematics by Thomas Heath. A notable feature of this section is a run of items highlighting non-European contributions to the development of geometry, especially one by the author of *The Crest of the Peacock*, George Gheverghese Joseph. It is rounded off with a detailed review of nineteenth century geometry by Gaston Darboux, penned at the turn of the century without the benefit that a time lapse often affords. The reader may judge whether Hardy is harsh in suggesting in his article that Darboux was a great geometer with little feel for the nature of his discipline. The section on the teaching of geometry is introduced by a new and comprehensive essay from The Mathematical Association’s historian, Michael Price. It is followed by what Price describes as a ‘concise and stylish attack on Euclid from an advanced pure mathematical standpoint’ [2, p. 55]. Its author is Bertrand Russell, the year 1902, the very year that Teaching Committee came into existence.

There is a greater focus on elucidating elementary geometry in the other three parts of the book. Articles on Pythagoras’ Theorem are introduced by Janet Jagger, an educationalist and former Chair of Teaching Committee. Ron Knott casts his expert eye over the articles on the golden section. Incidentally, his peerless Fibonacci website is a treasury of information on the sequence and the associated ratio. Articles on recreational geometry are prefaced by the thoughts of Brian Bolt, whose numerous books on recreational mathematics are much in the vein of Martin Gardner. It is a real pleasure to see reproduced here an extract of an early *Mathematical Gazette* article on ‘rep-tiles’ (replicating figures in the plane) by another hugely influential recreational mathematician, Solomon Golomb. His enthusiastic followers will also be pleased to find among the Desert Island Theorems previously unpublished proofs of a tromino theorem.

Since 1942, the BBC has broadcast a popular weekly radio programme called *Desert Island Discs*. These days its guests are invited to select eight pieces of music that they would take with them to a desert island. They also nominate a favourite book though, presumably to guarantee variety week on week, the key works of the great religions and the works of Shakespeare are excluded. Thirdly, a luxury inanimate object of no practical use is selected. The choices made by guests are little more than devices to prompt them to discuss their lives.

For this book, the ‘desert island theme’ has been developed in a different direction and for a different reason. Imagine you are a mathematician, cast away...
on a desert island. You have no access to the modern world and little prospect of being reunited with the rest of humanity. Which geometrical theorem would you least wish to be without, in some sense; which perhaps are you truly glad was discovered. The focus is more on the theorem but might well have something of the personal too. No claim is made to novelty in this shift from desert island music to desert island mathematics. Those who have previously written or spoken on desert island mathematics include David Burges, Tony Crilly and Colin Fletcher and it is appropriate that their theorems should be included in this book.

Prompted by an early success in persuading Coxeter to offer his Desert Island Theorem – he actually took no persuading at all, such is his love of geometry – a number of eminent mathematicians and physicists, teachers and educationalists were invited to follow suit. They were asked to nominate an elementary, geometrical theorem, to try to keep to a maximum of about 500 words plus a diagram and to adopt where possible what might be termed a ‘popular science’ style. As you will see, some found the brief restrictive but the brief was always intended to guide rather than stifle. Only in a very small number of cases was the same theorem chosen by two contributors.

The response from those approached was overwhelmingly positive and enthusiastic. The Mathematical Association notes with gratitude the affection shown towards it by the community of mathematicians, especially in Britain and North America during the preparation of this book. Among those in higher education and research there would appear to be fulsome recognition of the valuable support and advice given to mathematics teachers by The Mathematical Association and similar organizations worldwide.

The informality and brevity of the contributor’s designations has raised the eyebrows of some who have previewed material in this book. Leaders in their fields usually need little introduction and certainly no puffing up. Indeed, what needs to be added to an author’s name tends to vary inversely with eminence. In the minds of thousands of students or former students, Tom Apostol is the author of the two best calculus texts of all time. It is fitting therefore to refer to him as a teacher of the calculus, especially in a book put together to celebrate a centenary of relevance to mathematics teachers. Further, in most cases, academic positions (though not institutions) have been omitted as have titles and other honours. This is an attempt – possibly a vain attempt, human nature being what it is – to focus on the Desert Island Theorems themselves rather than the eminence of the writer. Several of them are beautifully crafted. Where a contributor is or has been a servant of The Mathematical Association, this is noted especially for the benefit of members. Each castaway has either suggested the designation provided or else expressed contentment with it.
General Introduction

There are thirty Desert Island Theorems in total, arranged very loosely into five groups:

A. ancient Greek geometry
B. elementary Euclidean geometry (of the last four centuries)
C. advanced Euclidean
d. spherical geometry and topology
E. geometrical physics.

As a rule of thumb, the difficulty level of the theorems tends to increase as the book proceeds. Arguably, there is a move further and further from pure geometry at the same time. The vast majority of the theorems should be accessible to those who have studied mathematics to the age of eighteen and possess a willingness to wrestle with some of the geometrical arguments. Readers with an interest in the development of geometry will find much to savour.

If it can ever be said that a single person has pushed back the boundaries of a branch of mathematics, particularly in recent times, then it might justifiably be said of Coxeter in relation to geometry. Coxeter’s sixty-six year association with the University of Toronto may have obfuscated the fact that he was born and raised in England and received his university education at Cambridge. The first recorded evidence of his prodigious grasp of geometry is to be found in the Mathematical Gazette, where in the issue of October 1926, the young ‘Donald’ Coxeter asked the readership via Alan Robson, his teacher at Marlborough School, if it knew of an ‘elementary verification’ of $\int_0^\pi \sec^{-1}(\sec x + 2) \, dx = \frac{5\pi}{24}$ and two similar results ‘suggested by a geometrical consideration and verified graphically’ [4]. At the time, Robson was already prominent in The Mathematical Association. He would go on to hold high office as Chair of Teaching Committee and as President, either side of the Second World War [2, pp. 151, 161]. The direct influence of Robson on Coxeter is undoubted, the indirect influence of the Mathematical Association and its Teaching Association rather more speculative.

Finally, let me return to my previous theme to consider the simple, economical, elegant and surprising geometry of the cauliflower. Almost any elementary treatment of the Fibonacci sequence and the golden section draws attention to the structure of the spiralling segments of the pineapple. Yet the cauliflower appears to have exactly the same structure and it seems a shame that this humble vegetable should have been so thoroughly eclipsed by an exotic fruit. If you look carefully enough the geometrical structure is certainly evident in the common white variety, though it is striking in the romanesco variety of Fibonacci’s homeland of northern Italy. A few years ago I was delighted to find that the cauliflower was not without its champion. Touring the Cité des Sciences et de l’Industrie at La Villette in Paris, I came across an exhibit which highlighted both the cauliflower’s spirals and
The spirals and fractals of the romanesco cauliflower
its self-replicating fractal form. But whether we invoke the pineapple, the nautilus shell, or Fibonacci’s romanesco cauliflower one thing seems clear – the attributes of the most attractive and pleasing geometry remain the same irrespective of whether the geometry is of our invention or of our discovery, crafted by man or designed by nature. In the words of John Keats: A thing of beauty is a joy for ever.

Solution to the Maps Question (by Doug French)
First, let $BA$ produced meet $XW$ at $B'$.

$$\angle BEX = \angle BB'X = \text{angle of rotation}$$

These angles are subtended by the same chord, $XB$. So $XBEB'$ is a cyclic quadrilateral.

Similarly, let $AD$ produced meet $WZ$ at $A'$.

$$\angle AEW = \angle AA'W.$$ 

These angles are subtended by the same chord, $AW$. So $WAEA'$ is a cyclic quadrilateral.

Since $E$ lies on the circumference of both circles, it is located at one of the points of intersection.

References

4. D. Coxeter (per A. R.), Note 853, *Mathematical Gazette* 13 (October 1926), 205. This was Coxeter’s first publication. [See Coxeter, H. S. M. *Twelve Geometry Essays* (Southern Illinois University Press, 1968) (Preface, p. vii).]