Elastic Wave Propagation
and Generation in
Seismology

Jose Pujol
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>Preface</td>
<td>xiii</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>Acknowledgements</td>
<td>xviii</td>
</tr>
<tr>
<td>1</td>
<td>Introduction to tensors and dyadics</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Summary of vector analysis</td>
<td>2</td>
</tr>
<tr>
<td>1.3</td>
<td>Rotation of Cartesian coordinates. Definition of a vector</td>
<td>7</td>
</tr>
<tr>
<td>1.4</td>
<td>Cartesian tensors</td>
<td>11</td>
</tr>
<tr>
<td>1.4.1</td>
<td>Tensor operations</td>
<td>14</td>
</tr>
<tr>
<td>1.4.2</td>
<td>Symmetric and anti-symmetric tensors</td>
<td>16</td>
</tr>
<tr>
<td>1.4.3</td>
<td>Differentiation of tensors</td>
<td>17</td>
</tr>
<tr>
<td>1.4.4</td>
<td>The permutation symbol</td>
<td>18</td>
</tr>
<tr>
<td>1.4.5</td>
<td>Applications and examples</td>
<td>19</td>
</tr>
<tr>
<td>1.4.6</td>
<td>Diagonalization of a symmetric second-order tensor</td>
<td>23</td>
</tr>
<tr>
<td>1.4.7</td>
<td>Isotropic tensors</td>
<td>28</td>
</tr>
<tr>
<td>1.4.8</td>
<td>Vector associated with a second-order anti-symmetric tensor</td>
<td>28</td>
</tr>
<tr>
<td>1.4.9</td>
<td>Divergence or Gauss’ theorem</td>
<td>29</td>
</tr>
<tr>
<td>1.5</td>
<td>Infinitesimal rotations</td>
<td>30</td>
</tr>
<tr>
<td>1.6</td>
<td>Dyads and dyadics</td>
<td>32</td>
</tr>
<tr>
<td>1.6.1</td>
<td>Dyads</td>
<td>33</td>
</tr>
<tr>
<td>1.6.2</td>
<td>Dyadics</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>Deformation. Strain and rotation tensors</td>
<td>40</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction</td>
<td>40</td>
</tr>
<tr>
<td>2.2</td>
<td>Description of motion. Lagrangian and Eulerian points of view</td>
<td>41</td>
</tr>
<tr>
<td>2.3</td>
<td>Finite strain tensors</td>
<td>43</td>
</tr>
<tr>
<td>2.4</td>
<td>The infinitesimal strain tensor</td>
<td>45</td>
</tr>
</tbody>
</table>
Contents

2.4.1 Geometric meaning of $\varepsilon_{ij}$ 46
2.4.2 Proof that $\varepsilon_{ij}$ is a tensor 49
2.5 The rotation tensor 50
2.6 Dyadic form of the strain and rotation tensors 51
2.7 Examples of simple strain fields 52

3 The stress tensor 59

3.1 Introduction 59
3.2 Additional continuum mechanics concepts 59
3.2.1 Example 63
3.3 The stress vector 64
3.4 The stress tensor 67
3.5 The equation of motion. Symmetry of the stress tensor 70
3.6 Principal directions of stress 72
3.7 Isotropic and deviatoric components of the stress tensor 72
3.8 Normal and shearing stress vectors 73
3.9 Stationary values and directions of the normal and shearing stress vectors 75
3.10 Mohr’s circles for stress 79

4 Linear elasticity – the elastic wave equation 84

4.1 Introduction 84
4.2 The equation of motion under the small-deformation approximation 85
4.3 Thermodynamical considerations 86
4.4 Strain energy 88
4.5 Linear elastic and hyperelastic deformations 90
4.6 Isotropic elastic solids 92
4.7 Strain energy density for the isotropic elastic solid 96
4.8 The elastic wave equation for a homogeneous isotropic medium 97

5 Scalar and elastic waves in unbounded media 100

5.1 Introduction 100
5.2 The 1-D scalar wave equation 100
5.2.1 Example 103
5.3 The 3-D scalar wave equation 103
5.4 Plane harmonic waves. Superposition principle 107
5.5 Spherical waves 111
5.6 Vector wave equation. Vector solutions 112
5.6.1 Properties of the Hansen vectors 115
## Contents

7.4.1 Homogeneous half-space 202  
7.4.2 Layer over a half-space. Dispersive Rayleigh waves 206  
7.4.3 Vertically heterogeneous medium 209  
7.5 Stoneley waves 212  
7.6 Propagation of dispersive waves 213  
7.6.1 Introductory example. The dispersive string 214  
7.6.2 Narrow-band waves. Phase and group velocity 215  
7.6.3 Broad-band waves. The method of stationary phase 220  
7.6.4 The Airy phase 227  

8 Ray theory 234  
8.1 Introduction 234  
8.2 Ray theory for the 3-D scalar wave equation 235  
8.3 Ray theory for the elastic wave equation 237  
  8.3.1 \( P \) and \( S \) waves in isotropic media 240  
8.4 Wave fronts and rays 242  
  8.4.1 Medium with constant velocity 244  
  8.4.2 Medium with a depth-dependent velocity 246  
  8.4.3 Medium with spherical symmetry 247  
8.5 Differential geometry of rays 248  
8.6 Calculus of variations. Fermat’s principle 254  
8.7 Ray amplitudes 258  
  8.7.1 Scalar wave equation 258  
  8.7.2 Elastic wave equation 261  
  8.7.3 Effect of discontinuities in the elastic parameters 268  
8.8 Examples 269  
  8.8.1 \( SH \) waves in a layer over a half-space at normal incidence 270  
  8.8.2 Ray theory synthetic seismograms 274  

9 Seismic point sources in unbounded homogeneous media 278  
9.1 Introduction 278  
9.2 The scalar wave equation with a source term 279  
9.3 Helmholtz decomposition of a vector field 281  
9.4 Lamé’s solution of the elastic wave equation 282  
9.5 The elastic wave equation with a concentrated force in the \( x_j \) direction 285  
  9.5.1 Type of motion 288  
  9.5.2 Near and far fields 289
### 9.5.3 Example. The far field of a point force at the origin in the $x_3$ direction 291

### 9.6 Green’s function for the elastic wave equation 295

### 9.7 The elastic wave equation with a concentrated force in an arbitrary direction 296

### 9.8 Concentrated couples and dipoles 297

### 9.9 Moment tensor sources. The far field 300

#### 9.9.1 Radiation patterns. $SV$ and $SH$ waves 303

### 9.10 Equivalence of a double couple and a pair of compressional and tensional dipoles 305

### 9.11 The tension and compression axes 306

### 9.12 Radiation patterns for the single couple $M_{31}$ and the double couple $M_{13} + M_{31}$ 308

### 9.13 Moment tensor sources. The total field 311

#### 9.13.1 Radiation patterns 313

### 10 The earthquake source in unbounded media 316

#### 10.1 Introduction 316

#### 10.2 A representation theorem 318

#### 10.3 Gauss’ theorem in the presence of a surface of discontinuity 321

#### 10.4 The body force equivalent to slip on a fault 322

#### 10.5 Slip on a horizontal plane. Point-source approximation. The double couple 325

#### 10.6 The seismic moment tensor 329

#### 10.7 Moment tensor for slip on a fault of arbitrary orientation 331

#### 10.8 Relations between the parameters of the conjugate planes 338

#### 10.9 Radiation patterns and focal mechanisms 339

#### 10.10 The total field. Static displacement 347

#### 10.11 Ray theory for the far field 352

### 11 Anelastic attenuation 357

#### 11.1 Introduction 357

#### 11.2 Harmonic motion. Free and damped oscillations 360

#### 11.2.1 Temporal $Q$ 362

#### 11.3 The string in a viscous medium 364

#### 11.4 The scalar wave equation with complex velocity 365

#### 11.4.1 Spatial $Q$ 366

#### 11.5 Attenuation of seismic waves in the Earth 367

#### 11.6 Mathematical aspects of causality and applications 370

#### 11.6.1 The Hilbert transform. Dispersion relations 371
Contents

11.6.2 Minimum-phase-shift functions 372
11.6.3 The Paley–Wiener theorem. Applications 375
11.7 Futterman’s relations 377
11.8 Kalinin and Azimi’s relation. The complex wave velocity 381
11.9 $t^*$ 384
11.10 The spectral ratio method. Window bias 384
11.11 Finely layered media and scattering attenuation 386

Hints 391

Appendices 407
A Introduction to the theory of distributions 407
B The Hilbert transform 419
C Green’s function for the 3-D scalar wave equation 422
D Proof of (9.5.12) 425
E Proof of (9.13.1) 428

Bibliography 431
Index 439
Introduction to tensors and dyadics

1.1 Introduction

Tensors play a fundamental role in theoretical physics. The reason for this is that physical laws written in tensor form are independent of the coordinate system used (Morse and Feshbach, 1953). Before elaborating on this point, consider a simple example, based on Segel (1977). Newton’s second law is \( \mathbf{f} = m \mathbf{a} \), where \( \mathbf{f} \) and \( \mathbf{a} \) are vectors representing the force and acceleration of an object of mass \( m \). This basic law does not have a coordinate system attached to it. To apply the law in a particular situation it will be convenient to select a coordinate system that simplifies the mathematics, but there is no question that any other system will be equally acceptable. Now consider an example from elasticity, discussed in Chapter 3. The stress vector \( \mathbf{T} \) (force/area) across a surface element in an elastic solid is related to the vector \( \mathbf{n} \) normal to the same surface via the stress tensor. The derivation of this relation is carried out using a tetrahedron with faces along the three coordinate planes in a Cartesian coordinate system. Therefore, it is reasonable to ask whether the same result would have been obtained if a different Cartesian coordinate system had been used, or if a spherical, or cylindrical, or any other curvilinear system, had been used. Take another example. The elastic wave equation will be derived in a Cartesian coordinate system. As discussed in Chapter 4, two equations will be found, one in component form and one in vector form in terms of a combination of gradient, divergence, and curl. Again, here there are some pertinent questions regarding coordinate systems. For example, can either of the two equations be applied in non-Cartesian coordinate systems? The reader may already know that only the latter equation is generally applicable, but may not be aware that there is a mathematical justification for that fact, namely, that the gradient, divergence, and curl are independent of the coordinate system (Morse and Feshbach, 1953). These questions are generally not discussed in introductory texts, with the consequence that the reader fails to grasp the deeper meaning of the concepts of
Introduction to tensors and dyadics

vectors and tensors. It is only when one realizes that physical entities (such as force, acceleration, stress tensor, and so on) and the relations among them have an existence independent of coordinate systems, that it is possible to appreciate that there is more to tensors than what is usually discussed. It is possible, however, to go through the basic principles of stress and strain without getting into the details of tensor analysis. Therefore, some parts of this chapter are not essential for the rest of the book.

Tensor analysis, in its broadest sense, is concerned with arbitrary curvilinear coordinates. A more restricted approach concentrates on orthogonal curvilinear coordinates, such as cylindrical and spherical coordinates. These coordinate systems have the property that the unit vectors at a given point in space are perpendicular (i.e. orthogonal) to each other. Finally, we have the rectangular Cartesian system, which is also orthogonal. The main difference between general orthogonal and Cartesian systems is that in the latter the unit vectors do not change as a function of position, while this is not true in the former. Unit vectors for the spherical system will be given in §9.9.1. The theory of tensors in non-Cartesian systems is exceedingly complicated, and for this reason we will limit our study to Cartesian tensors. However, some of the most important relations will be written using dyadics (see §1.6), which provide a symbolic representation of tensors independent of the coordinate system. It may be useful to note that there are oblique Cartesian coordinate systems of importance in crystallography, for example, but in the following we will consider the rectangular Cartesian systems only.

1.2 Summary of vector analysis

It is assumed that the reader is familiar with the material summarized in this section (see, e.g., Lass, 1950; Davis and Snider, 1991).

A vector is defined as a directed line segment, having both magnitude and direction. The magnitude, or length, of a vector \( \mathbf{a} \) will be represented by \(|\mathbf{a}|\). The sum and the difference of two vectors, and the multiplication of a vector by a scalar (real number) are defined using geometric rules. Given two vectors \( \mathbf{a} \) and \( \mathbf{b} \), two products between them have been defined.

Scalar, or dot, product:

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \alpha, \quad (1.2.1)
\]

where \( \alpha \) is the angle between the vectors.

Vector, or cross, product:

\[
\mathbf{a} \times \mathbf{b} = (|\mathbf{a}| |\mathbf{b}| \sin \alpha) \mathbf{n}, \quad (1.2.2)
\]
1.2 Summary of vector analysis

Fig. 1.1. Rectangular Cartesian coordinate system with unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and decomposition of an arbitrary vector $\mathbf{v}$ into components $v_1, v_2, v_3$.

where $\alpha$ is as before, and $\mathbf{n}$ is a unit vector (its length is equal to 1) perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ such that the three vectors form a right-handed system.

An important property of the vector product, derived using geometric arguments, is the distributive law

$$ (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. $$

(1.2.3)

By introducing a rectangular Cartesian coordinate system it is possible to write a vector in terms of three components. Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ be the three unit vectors along the $x_1$, $x_2$, and $x_3$ axes of Fig. 1.1. Then any vector $\mathbf{v}$ can be written as

$$ \mathbf{v} = (v_1, v_2, v_3) = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3 = \sum_{i=1}^{3} v_i \mathbf{e}_i. $$

(1.2.4)

The components $v_1$, $v_2$, and $v_3$ are the orthogonal projections of $\mathbf{v}$ in the directions of the three axes (Fig. 1.1).

Before proceeding, a few words concerning the notation are necessary. A vector will be denoted by a bold-face letter, while its components will be denoted by the same letter in italics with subindices (literal or numerical). A bold-face letter with a subindex represents a vector, not a vector component. The three unit vectors defined above are examples of the latter. If we want to write the $k$th component of the unit vector $\mathbf{e}_j$ we will write $(\mathbf{e}_j)_k$. For example, $(\mathbf{e}_2)_1 = 0$, $(\mathbf{e}_2)_2 = 1$, and $(\mathbf{e}_2)_3 = 0$. In addition, although vectors will usually be written in row form (e.g., as in (1.2.4)), when they are involved in matrix operations they should be considered as column vectors, i.e., as matrices of one column and three rows. For example, the matrix form of the scalar product $\mathbf{a} \cdot \mathbf{b}$ is $\mathbf{a}^T \mathbf{b}$, where T indicates transposition.
Introduction to tensors and dyadics

When the scalar product is applied to the unit vectors we find

\[ e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0 \]  
\[ (1.2.5) \]

\[ e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1. \]  
\[ (1.2.6) \]

Equations (1.2.5) and (1.2.6) can be summarized as follows:

\[ e_i \cdot e_j = \delta_{ij} = \begin{cases} 
1, & i = j \\
0, & i \neq j. 
\end{cases} \]  
\[ (1.2.7) \]

The symbol \( \delta_{jk} \) is known as the Kronecker delta, which is an example of a second-order tensor, and will play an important role in this book. As an example of (1.2.7), \( e_2 \cdot e_k \) is zero unless \( k = 2 \), in which case the scalar product is equal to 1.

Next we derive an alternative expression for a vector \( v \). Using (1.2.4), the scalar product of \( v \) and \( e_i \) is

\[ v \cdot e_i = \left( \sum_{k=1}^{3} v_k e_k \right) \cdot e_i = \sum_{k=1}^{3} v_k e_k \cdot e_i = \sum_{k=1}^{3} v_k (e_k \cdot e_i) = v_i. \]  
\[ (1.2.8) \]

Note that when applying (1.2.4) the subindex in the summation must be different from \( i \). To obtain (1.2.8) the following were used: the distributive law of the scalar product, the law of the product by a scalar, and (1.2.7). Equation (1.2.8) shows that the \( i \)th component of \( v \) can be written as

\[ v_i = v \cdot e_i. \]  
\[ (1.2.9) \]

When (1.2.9) is introduced in (1.2.4) we find

\[ v = \sum_{i=1}^{3} (v \cdot e_i) e_i. \]  
\[ (1.2.10) \]

This expression will be used in the discussion of dyadics (see §1.6).

In terms of its components the length of the vector is given by

\[ |v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = (v \cdot v)^{1/2}. \]  
\[ (1.2.11) \]

Using purely geometric arguments it is found that the scalar and vector products can be written in component form as follows:

\[ u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 \]  
\[ (1.2.12) \]

and

\[ u \times v = (u_2 v_3 - u_3 v_2) e_1 + (u_3 v_1 - u_1 v_3) e_2 + (u_1 v_2 - u_2 v_1) e_3. \]  
\[ (1.2.13) \]

The last expression is based on the use of (1.2.3).
1.2 Summary of vector analysis

Vectors, and vector operations such as the scalar and vector products, among others, are defined independently of any coordinate system. Vector relations derived without recourse to vector components will be valid when written in component form regardless of the coordinate system used. Of course, the same vector may (and generally will) have different components in different coordinate systems, but they will represent the same geometric entity. This is true for Cartesian and more general coordinate systems, such as spherical and cylindrical ones, but in the following we will consider the former only.

Now suppose that we want to define new vector entities based on operations on the components of other vectors. In view of the comments in §1.1 it is reasonable to expect that not every arbitrary definition will represent a vector, i.e., an entity intrinsically independent of the coordinate system used to represent the space. To see this consider the following example, which for simplicity refers to vectors in two-dimensional (2-D) space. Given a vector \( \mathbf{u} = (u_1, u_2) \), define a new vector \( \mathbf{v} = (u_1 + \lambda, u_2 + \lambda) \), where \( \lambda \) is a nonzero scalar. Does this definition result in a vector? To answer this question draw the vectors \( \mathbf{u} \) and \( \mathbf{v} \) (Fig. 1.2a), rotate the original coordinate axes, decompose \( \mathbf{u} \) into its new components \( u'_1 \) and \( u'_2 \), add \( \lambda \) to each of them, and draw the new vector \( \mathbf{v}' = (u'_1 + \lambda, u'_2 + \lambda) \). Clearly, \( \mathbf{v} \) and \( \mathbf{v}' \) are not the same geometric object. Therefore, our definition does not represent a vector.

Now consider the following definition: \( \mathbf{v} = (\lambda u_1, \lambda u_2) \). After a rotation similar to the previous one we see that \( \mathbf{v} = \mathbf{v}' \) (Fig. 1.2b), which is not surprising, as this definition corresponds to the multiplication of a vector by a scalar.

Let us look now at a more complicated example. Suppose that given two vectors \( \mathbf{u} \) and \( \mathbf{v} \) we want to define a third vector \( \mathbf{w} \) as follows:

\[
\mathbf{w} = (u_2 v_3 + u_3 v_2)\mathbf{e}_1 + (u_3 v_1 + u_1 v_3)\mathbf{e}_2 + (u_1 v_2 + u_2 v_1)\mathbf{e}_3. \tag{1.2.14}
\]

Note that the only difference with the vector product (see (1.2.13)) is the replacement of the minus signs by plus signs. As before, the question is whether this definition is independent of the coordinate system. In this case, however, finding an answer is not straightforward. What one should do is to compute the components \( w_1, w_2, w_3 \) in the original coordinate system, draw \( \mathbf{w} \), perform a rotation of axes, find the new components of \( \mathbf{u} \) and \( \mathbf{v} \), compute \( w'_1, w'_2, \) and \( w'_3 \), draw \( \mathbf{w}' \) and compare it with \( \mathbf{w} \). If it is found that the two vectors are different, then it is obvious that (1.2.14) does not define a vector. If the two vectors are equal it might be tempting to say that (1.2.14) does indeed define a vector, but this conclusion would not be correct because there may be other rotations for which \( \mathbf{w} \) and \( \mathbf{w}' \) are not equal.

These examples should convince the reader that establishing the vectorial character of an entity defined by its components requires a definition of a vector that will take this question into account automatically. Only then will it be possible to
Introduction to tensors and dyadics

Fig. 1.2. (a) Vectors $v$ and $v'$ obtained from a vector $u$ as follows. For $v$, add a constant $\lambda$ to the components $u_1$ and $u_2$. For $v'$, add a constant $\lambda$ to the components $u'_1$ and $u'_2$ obtained by rotation of the axis. Because $v$ and $v'$ are not the same vector, we can conclude that the entity obtained by adding a constant to the components of a vector does constitute a vector under a rotation of coordinates. (b) Similar to the construction above, but with the constant $\lambda$ multiplying the vector components. In this case $v$ and $v'$ coincide, which agrees with the fact that the operation defined is just the multiplication of a vector by a scalar. After Santalo (1969).

answer the previous question in a general way. However, before introducing the new definition it is necessary to study coordinate rotations in some more detail. This is done next.
1.3 Rotation of Cartesian coordinates. Definition of a vector

Let $Ox_1$, $Ox_2$, and $Ox_3$ represent a Cartesian coordinate system and $Ox'_1$, $Ox'_2$, $Ox'_3$ another system obtained from the previous one by a rotation about their common origin $O$ (Fig. 1.3). Let $e_1$, $e_2$, and $e_3$ and $e'_1$, $e'_2$, and $e'_3$ be the unit vectors along the three axes in the original and rotated systems. Finally, let $a_{ij}$ denote the cosine of the angle between $Ox'_i$ and $Ox_j$. The $a_{ij}$'s are known as direction cosines, and are related to $e'_i$ and $e_j$ by

$$e'_i \cdot e_j = a_{ij}.$$  

(1.3.1)

Given an arbitrary vector $v$ with components $v_1$, $v_2$, and $v_3$ in the original system, we are interested in finding the components $v'_1$, $v'_2$, and $v'_3$ in the rotated system. To find the relation between the two sets of components we will consider first the relation between the corresponding unit vectors. Using (1.3.1) $e'_i$ can be written as

$$e'_i = a_{i1}e_1 + a_{i2}e_2 + a_{i3}e_3 = \sum_{j=1}^{3} a_{ij}e_j$$  

(1.3.2)

(Problem 1.3a).

Fig. 1.3. Rotation of axes. Primed and unprimed quantities refer to the original and rotated coordinate systems, respectively. Both systems are rectangular Cartesian. The quantities $a_{ij}$ indicate the scalar product $e'_i \cdot e_j$. The vector $v$ exists independent of the coordinate system. Three relevant angles are shown.
Furthermore, in the original and rotated systems $\mathbf{v}$ can be written as

$$
\mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j
$$

(1.3.3)

and

$$
\mathbf{v} = \sum_{i=1}^{3} v'_i \mathbf{e}'_i.
$$

(1.3.4)

Now introduce (1.3.2) in (1.3.4)

$$
\mathbf{v} = \sum_{i=1}^{3} v'_i \sum_{j=1}^{3} a_{ij} \mathbf{e}_j \equiv \sum_{j=1}^{3} \left( \sum_{i=1}^{3} a_{ij} v'_i \right) \mathbf{e}_j.
$$

(1.3.5)

Since (1.3.3) and (1.3.5) represent the same vector, and the three unit vectors $\mathbf{e}_1$, $\mathbf{e}_1$, and $\mathbf{e}_3$ are independent of each other, we conclude that

$$
v_j = \sum_{i=1}^{3} a_{ij} v'_i.
$$

(1.3.6)

If we write the $\mathbf{e}_j$s in terms of the $\mathbf{e}'_i$s and replace them in (1.3.3) we find that

$$
v'_i = \sum_{j=1}^{3} a_{ij} v_j
$$

(1.3.7)

(Problem 1.3b).

Note that in (1.3.6) the sum is over the first subindex of $a_{ij}$, while in (1.3.7) the sum is over the second subindex of $a_{ij}$. This distinction is critical and must be respected.

Now we are ready to introduce the following definition of a vector:

three scalars are the components of a vector if under a rotation of coordinates they transform according to (1.3.7).

What this definition means is that if we want to define a vector by some set of rules, we have to verify that the vector components satisfy the transformation equations.

Before proceeding we will introduce a summation convention (due to Einstein) that will simplify the mathematical manipulations significantly. The convention applies to monomial expressions (such as a single term in an equation) and consists of dropping the sum symbol and summing over repeated indices. This convention requires that the same index should appear no more than twice in the same term.

1 In this book the convention will not be applied to uppercase indices
1.3 Rotation of Cartesian coordinates. Definition of a vector

Repeated indices are known as *dummy indices*, while those that are not repeated are called *free indices*. Using this convention, we will write, for example,

\[ \mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e}_j = v_j \mathbf{e}_j \]

(1.3.8)

\[ v_j = \sum_{i=1}^{3} a_{ij} v_i' = a_{ij} v_i' \]

(1.3.9)

\[ v_i' = \sum_{j=1}^{3} a_{ij} v_j = a_{ij} v_j. \]

(1.3.10)

It is important to have a clear idea of the difference between free and dummy indices. A particular dummy index can be changed at will as long as it is replaced (in its two occurrences) by some other index not equal to any other existing indices in the same term. Free indices, on the other hand, are fixed and cannot be changed inside a single term. However, a free index can be replaced by another as long as the change is effected in all the terms in an equation, and the new index is different from all the other indices in the equation. In (1.3.9) \( i \) is a dummy index and \( j \) is a free index, while in (1.3.10) their role is reversed. The examples below show legal and illegal index manipulations.

The following relations, derived from (1.3.9), are true

\[ v_j = a_{ij} v_i' = a_{kj} v_k' = a_{ij} v_i' \]

(1.3.11)

because the repeated index \( i \) was replaced by a different repeated index (equal to \( k \) or \( l \)). However, it would not be correct to replace \( i \) by \( j \) because \( j \) is already present in the equation. If \( i \) were replaced by \( j \) we would have

\[ v_j = a_{ij} v_j, \]

(1.3.12)

which would not be correct because the index \( j \) appears more than twice in the right-hand term, which is not allowed. Neither would it be correct to write

\[ v_j = a_{ik} v_j' \]

(1.3.13)

because the free index \( j \) has been changed to \( k \) only in the right-hand term. On the other hand, (1.3.9) can be written as

\[ v_k = a_{ik} v_i' \]

(1.3.14)

because the free index \( j \) has been replaced by \( k \) on both sides of the equation.

As (1.3.9) and (1.3.10) are of fundamental importance, it is necessary to pay attention to the fact that in the former the sum is over the first index of \( a_{ij} \) while
in the latter the sum is over the second index of \( a_{ij} \). Also note that (1.3.10) can be written as the product of a matrix and a vector:

\[
\mathbf{v}' = \begin{pmatrix} v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{A} \mathbf{v}, \quad (1.3.15)
\]

where \( \mathbf{A} \) is the matrix with elements \( a_{ij} \).

It is clear that (1.3.9) can be written as

\[
\mathbf{v} = \mathbf{A}^T \mathbf{v}', \quad (1.3.16)
\]

where the superscript \( T \) indicates transposition.

Now we will derive an important property of \( \mathbf{A} \). By introducing (1.3.10) in (1.3.9) we obtain

\[
v_j = a_{ij} a_{ik} v_k. \quad (1.3.17)
\]

Note that it was necessary to change the dummy index in (1.3.10) to satisfy the summation convention. Equation (1.3.17) implies that any of the three components of \( \mathbf{v} \) is a combination of all three components. However, this cannot be generally true because \( \mathbf{v} \) is an arbitrary vector. Therefore, the right-hand side of (1.3.17) must be equal to \( v_j \), which in turn implies that the product \( a_{ij} a_{ik} \) must be equal to unity when \( j = k \), and equal to zero when \( j \neq k \). This happens to be the definition of the Kronecker delta \( \delta_{jk} \) introduced in (1.2.7), so that

\[
a_{ij} a_{ik} = \delta_{jk}. \quad (1.3.18)
\]

If (1.3.9) is introduced in (1.3.10) we obtain

\[
a_{ij} a_{kj} = \delta_{ik}. \quad (1.3.19)
\]

Setting \( i = k \) in (1.3.19) and writing in full gives

\[
1 = a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = |\mathbf{e}_i'|^2; \quad i = 1, 2, 3, \quad (1.3.20)
\]

where the equality on the right-hand side follows from (1.3.2).

When \( i \neq k \), (1.3.19) gives

\[
0 = a_{i1} a_{k1} + a_{i2} a_{k2} + a_{i3} a_{k3} = \mathbf{e}_i' \cdot \mathbf{e}_k', \quad (1.3.21)
\]

where the equality on the right-hand side also follows from (1.3.2). Therefore, (1.3.19) summarizes the fact that the \( \mathbf{e}_j' \)s are unit vectors orthogonal to each other, while (1.3.18) does the same thing for the \( \mathbf{e}_i \)s. Any set of vectors having these properties is known as an orthonormal set.
In matrix form, (1.3.18) and (1.3.19) can be written as

\[ A^T A = AA^T = I, \]  

(1.3.22)

where \( I \) is the identity matrix.

Equation (1.3.22) can be rewritten in the following useful way:

\[ A^T = A^{-1}; \quad (A^T)^{-1} = A, \]  

(1.3.23)

where the superscript \(-1\) indicates matrix inversion. From (1.3.22) we also find

\[ |AA^T| = |A||A^T| = |A|^2 = |I| = 1, \]  

(1.3.24)

where vertical bars indicate the determinant of a matrix.

Linear transformations with a matrix such that its determinant squared is equal to 1 are known as orthogonal transformations. When \(|A| = 1\), the transformation corresponds to a rotation. When \(|A| = -1\), the transformation involves the reflection of one coordinate axis in a coordinate plane. An example of reflection is the transformation that leaves the \(x_1\) and \(x_2\) axes unchanged and replaces the \(x_3\) axis by \(-x_3\). Reflections change the orientation of the space: if the original system is right-handed, then the new system is left-handed, and vice versa.

### 1.4 Cartesian tensors

In subsequent chapters the following three tensors will be introduced.

1. The strain tensor \( \varepsilon_{ij} \):

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right); \quad i, j = 1, 2, 3, \]  

(1.4.1)

where the vector \( \mathbf{u} = (u_1, u_2, u_3) \) is the displacement suffered by a particle inside a body when it is deformed.

2. The stress tensor \( \tau_{ij} \):

\[ T_i = \tau_{ij} n_j; \quad i = 1, 2, 3, \]  

(1.4.2)

where \( T_i \) and \( n_j \) indicate the components of the stress vector and normal vector referred to in §1.1.

3. The elastic tensor \( c_{ijkl} \), which relates stress to strain:

\[ \tau_{ij} = c_{ijkl} \varepsilon_{kl}. \]  

(1.4.3)

Let us list some of the differences between vectors and tensors. First, while a vector can be represented by a single symbol, such as \( \mathbf{u} \), or by its components, such as \( u_j \), a tensor can only be represented by its components (e.g., \( \varepsilon_{ij} \)), although the introduction of dyadics (see §1.6) will allow the representation of tensors by...
Introduction to tensors and dyadics

single symbols. Secondly, while vector components carry only one subindex, tensors carry two subindices or more. Thirdly, in the three-dimensional space we are considering, a vector has three components, while \( \varepsilon_{ij} \) and \( \tau_{ij} \) have \( 3 \times 3 \), or nine, components, and \( c_{ijkl} \) has 81 components \( (3 \times 3 \times 3 \times 3) \). Tensors \( \varepsilon_{ij} \) and \( \tau_{ij} \) are known as second-order tensors, while \( c_{ijkl} \) is a fourth-order tensor, with the order of the tensor being given by the number of free indices. There are also differences among the tensors shown above. For example, \( \varepsilon_{ij} \) is defined in terms of operations (derivatives) on the components of a single vector, while \( \tau_{ij} \) appears in a relation between two vectors. \( c_{ijkl} \), on the other hand, relates two tensors.

Clearly, tensors offer more variety than vectors, and because they are defined in terms of components, the comments made in connection with vector components and the rotation of axes also apply to tensors. To motivate the following definition of a second-order tensor consider the relation represented by (1.4.2). For this relation to be independent of the coordinate system, upon a rotation of axes we must have

\[
T'_l = \tau'_{lk} n'_k. \tag{1.4.4}
\]

In other words, the functional form of the relation must remain the same after a change of coordinates. We want to find the relation between \( \tau'_{lk} \) and \( \tau_{ij} \) that satisfies (1.4.2) and (1.4.4). To do that multiply (1.4.2) by \( a_{li} \) and sum over \( i \):

\[
a_{li} T_i = a_{li} \tau_{ij} n_j. \tag{1.4.5}
\]

Before proceeding rewrite \( T'_i \) and \( n_j \) using (1.3.10) with \( v \) replaced by \( T \) and (1.3.9) with \( v \) replaced by \( n \). This gives

\[
T'_i = a_{li} T_i; \quad n_j = a_{kj} n'_k. \tag{1.4.6a,b}
\]

From (1.4.6a,b), (1.4.2), and (1.4.5) we find

\[
T'_i = a_{li} T_i = a_{li} \tau_{ij} n_j = a_{li} \tau_{ij} a_{kj} n'_k = (a_{li} a_{kj} \tau_{ij}) n'_k. \tag{1.4.7}
\]

Now subtracting (1.4.4) from (1.4.7) gives

\[
0 = (\tau'_{lk} - a_{li} a_{kj} \tau_{ij}) n'_k. \tag{1.4.8}
\]

As \( n_k \) is an arbitrary vector, the factor in parentheses in (1.4.8) must be equal to zero (Problem 1.7), so that

\[
\tau'_{lk} = a_{li} a_{kj} \tau_{ij}. \tag{1.4.9}
\]

Note that (1.4.9) does not depend on the physical nature of the quantities involved in (1.4.2). Only the functional relation matters. This result motivates the following definition.
1.4 Cartesian tensors

Second-order tensor. Given nine quantities \( t_{ij} \), they constitute the components of a second-order tensor if they transform according to

\[
t'_{ij} = a_{il} a_{jk} t_{lk}
\]

under a change of coordinates \( v'_i = a_{ij} v_j \).

To write the tensor components in the unprimed system in terms of the components in the primed system, multiply (1.4.10) by \( a_{im} a_{jn} \) and sum over \( i \) and \( j \), and use the orthogonality relation (1.3.18):

\[
a_{im} a_{jn} t'_{ij} = a_{im} a_{jn} a_{il} a_{jk} t_{lk} = a_{im} a_{jn} a_{il} a_{jk} t_{lk} = \delta_{lm} \delta_{kn} t_{lk} = t_{mn}.
\]

(1.4.11)

Therefore,

\[
t_{mn} = a_{im} a_{jn} t'_{ij}.
\]

(1.4.12)

As (1.4.10) and (1.4.12) are similar, it is important to make sure that the arrangement of indices is strictly adhered to.

Equation (1.4.11) illustrates an important aspect of the Kronecker delta. For a given \( m \) and \( n \), the expression \( \delta_{lm} \delta_{kn} t_{lk} \) is a double sum over \( l \) and \( k \), so that there are nine terms. However, since \( \delta_{lm} \) and \( \delta_{kn} \) are equal to zero except when \( l = m \) and \( k = n \), in which case the deltas are equal to one, the only nonzero term in the sum is \( t_{mn} \). Therefore, the equality \( \delta_{lm} \delta_{kn} t_{lk} = t_{mn} \) can be derived formally by replacing \( l \) and \( k \) in \( t_{lk} \) by \( m \) and \( n \) and by dropping the deltas.

The extension of (1.4.10) to higher-order tensors is straightforward.

Tensor of order \( n \). Given \( 3^n \) quantities \( t_{i_1 i_2 \ldots i_n} \), they constitute the components of a tensor of order \( n \) if they transform according to

\[
t'_{i_1 i_2 \ldots i_n} = a_{i_1 j_1} a_{i_2 j_2} \ldots a_{i_n j_n} t_{j_1 j_2 \ldots j_n}
\]

under a change of coordinates \( v'_i = a_{ij} v_j \). All the indices \( i_1, i_2, \ldots \) and \( j_1, j_2, \ldots \) can be 1, 2, 3.

The extension of (1.4.12) to tensors of order \( n \) is obvious. For example, for a third-order tensor we have the following relations:

\[
t'_{ijk} = a_{ij} a_{jm} a_{kn} t_{lmn}; \quad t_{mnp} = a_{im} a_{jn} a_{kp} t'_{ijk}.
\]

(1.4.14a,b)

The definition of tensors can be extended to include vectors and scalars, which can be considered as tensors of orders one and zero, respectively. The corresponding number of free indices are one and zero, but this does not mean that dummy indices are not allowed (such as in \( t_{jj} \) and \( u_{i1} \), introduced below).

An important consequence of definitions (1.4.10) and (1.4.13) is that if the components of a tensor are all equal to zero in a given coordinate system, they will be equal to zero in any other coordinate system.
In the following we will indicate vectors and tensors by their individual components. For example, $u_i$, $u_j$, and $u_m$, etc., represent the same vector $\mathbf{u}$, while $t_{ij}$, $t_{ik}$, $t_{mn}$, etc., represent the same tensor. As noted earlier, the introduction of dyadics will afford the representation of tensors by single symbols, but for the special case of a second-order tensor we can associate it with a $3 \times 3$ matrix. For example, the matrix $T$ corresponding to the tensor $t_{ij}$ is given by

$$
T = \begin{pmatrix}
  t_{11} & t_{12} & t_{13} \\
  t_{21} & t_{22} & t_{23} \\
  t_{31} & t_{32} & t_{33}
\end{pmatrix}.
$$

(1.4.15)

Introduction of $T$ is convenient because (1.4.10), which can be written as

$$
t'_{ij} = a_{il} t_{lk} a_{jk},
$$

(1.4.16)

has a simple expression in matrix form:

$$
T' = A T A^T,
$$

(1.4.17)

where $A$ is the matrix introduced in (1.3.15) and

$$
T' = \begin{pmatrix}
  t'_{11} & t'_{12} & t'_{13} \\
  t'_{21} & t'_{22} & t'_{23} \\
  t'_{31} & t'_{32} & t'_{33}
\end{pmatrix}.
$$

(1.4.18)

Equation (1.4.17) is very useful in actual computations.

To express $T$ in terms of $T'$, write (1.4.12) as

$$
t_{mn} = a_{im} t'_{ij} a_{ja},
$$

(1.4.19)

which gives

$$
T = A^T T' A.
$$

(1.4.20)

Otherwise, solve (1.4.17) for $T$ by multiplying both sides by $(A^T)^{-1}$ on the right and by $A^{-1}$ on the left and use (1.3.23).

### 1.4.1 Tensor operations

1. **Addition or subtraction of two tensors.** The result is a new tensor whose components are the sum or difference of the corresponding components. These two operations are defined for tensors of the same order. As an example, given the tensors $t_{ij}$ and $s_{ij}$, their sum or difference is the tensor $b_{ij}$ with components

$$
b_{ij} = t_{ij} \pm s_{ij}.
$$

(1.4.21)

Let us verify that $b_{ij}$ is indeed a tensor. This requires writing the components
1.4 Cartesian tensors

of \( t_{ij} \) and \( s_{ij} \) in the primed system, adding or subtracting them together and verifying that (1.4.10) is satisfied. From (1.4.10) we obtain

\[
    t'_{ij} = a_{il}a_{jm}t_{lm} \tag{1.4.22}
\]

\[
    s'_{ij} = a_{il}a_{jm}s_{lm}. \tag{1.4.23}
\]

Adding or subtracting (1.4.22) and (1.4.23) gives

\[
    b'_{ij} = t'_{ij} \pm s'_{ij} = a_{il}a_{jm}(t_{lm} \pm s_{lm}) = a_{ij}a_{jm}b_{lm}. \tag{1.4.24}
\]

Equation (1.4.24) shows that \( b_{ij} \) transforms according to (1.4.10) and is therefore a tensor.

(2) Multiplication of a tensor by a scalar. Each component of the tensor is multiplied by the scalar. The result of this operation is a new tensor. For example, multiplication of \( t_{ij} \) by a scalar \( \lambda \) gives the tensor \( b_{ij} \) with components

\[
    b_{ij} = \lambda t_{ij}. \tag{1.4.25}
\]

To show that \( b_{ij} \) is a tensor we proceed as before

\[
    b'_{ij} = \lambda t'_{ij} = \lambda a_{il}a_{jm}t_{lm} = a_{ij}a_{jm}(\lambda t_{lm}) = a_{ij}a_{jm}b_{lm}. \tag{1.4.26}
\]

(3) Outer product of two tensors. This gives a new tensor whose order is equal to the sum of the orders of the two tensors and whose components are obtained by multiplication of the components of the two tensors. From this definition it should be clear that the indices in one tensor cannot appear among the indices in the other tensor. As an example, consider the outer product of \( t_{ij} \) and \( u_k \).

The result is the tensor \( s_{ijk} \) given by

\[
    s_{ijk} = t_{ij}u_k. \tag{1.4.27}
\]

To show that \( s_{ijk} \) is a tensor proceed as before

\[
    s'_{ijk} = t'_{ij}u'_k = a_{il}a_{jm}a_{kn}u_n = a_{ij}a_{jm}a_{kn}u_n = a_{ij}a_{jm}a_{kn}s_{lmn}. \tag{1.4.28}
\]

As another example, the outer product of two vectors \( a \) and \( b \) is the tensor with components \( a_i b_j \). This particular outer product will be considered again when discussing dyadics.

(4) Contraction of indices. In a tensor of order two or higher, set two indices equal to each other. As the two contracted indices become dummy indices, they indicate a sum over the repeated index. By contraction, the order of the tensor is reduced by two. For example, given \( t_{ij} \), by contraction of \( i \) and \( j \) we obtain the scalar

\[
    t_{ii} = t_{11} + t_{22} + t_{33}, \tag{1.4.29}
\]

which is known as the trace of \( t_{ij} \). Note that when \( t_{ij} \) is represented by a matrix,
Introduction to tensors and dyadics

t_{ij} corresponds to the sum of its diagonal elements, which is generally known as the trace of the matrix. Another example of contraction is the divergence of a vector, discussed in §1.4.5.

(5) Inner product, or contraction, of two tensors. Given two tensors, first form their outer product and then apply a contraction of indices using one index from each tensor. For example, the scalar product of two vectors \( \mathbf{a} \) and \( \mathbf{b} \) is equal to their inner product, given by \( a_i b_i \). We will refer to the inner product as the contraction of two tensors. By extension, a product involving \( a_{ij} \), as in (1.4.5) and (1.4.11), for example, will also be called a contraction.

### 1.4.2 Symmetric and anti-symmetric tensors

A second-order tensor \( t_{ij} \) is symmetric if

\[
t_{ij} = t_{ji}
\]

and anti-symmetric if

\[
t_{ij} = -t_{ji}.
\]

Any second-order tensor \( b_{ij} \) can be written as the following identity:

\[
b_{ij} \equiv \frac{1}{2} b_{ij} + \frac{1}{2} b_{ji} - \frac{1}{2} b_{ji} = \frac{1}{2} (b_{ij} + b_{ji}) + \frac{1}{2} (b_{ij} - b_{ji}).
\]

Clearly, the tensors in parentheses are symmetric and anti-symmetric, respectively. Therefore, \( b_{ij} \) can be written as

\[
b_{ij} = s_{ij} + a_{ij}
\]

with

\[
s_{ij} = s_{ji} = \frac{1}{2} (b_{ij} + b_{ji})
\]

and

\[
a_{ij} = -a_{ji} = \frac{1}{2} (b_{ij} - b_{ji}).
\]

Examples of symmetric second-order tensors are the Kronecker delta, the strain tensor (as can be seen from (1.4.1)), and the stress tensor, as will be shown in the next chapter.

For higher-order tensors, symmetry and anti-symmetry are referred to pairs of indices. A tensor is completely symmetric (anti-symmetric) if it is symmetric (anti-symmetric) for all pairs of indices. For example, if \( t_{ijk} \) is completely symmetric, then

\[
t_{ijk} = t_{jik} = t_{ikj} = t_{kij} = t_{kji} = t_{jki}.
\]
If $t_{ijk}$ is completely anti-symmetric, then

$$t_{ijk} = -t_{jik} = -t_{kij} = t_{kji} = t_{jki}.$$  \hfill (1.4.37)

The permutation symbol, introduced in §1.4.4, is an example of a completely anti-symmetric entity.

The elastic tensor in (1.4.3) has symmetry properties different from those described above. They will be described in detail in Chapter 4.

### 1.4.3 Differentiation of tensors

Let $t_{ij}$ be a function of the coordinates $x_i \ (i = 1, 2, 3)$. From (1.4.10) we know that

$$t'_{ij} = a_{ik}a_{jl}t_{kl}. \hfill (1.4.38)$$

Now differentiate both sides of (1.4.38) with respect to $x'_m$,

$$\frac{\partial t'_{ij}}{\partial x'_m} = a_{ik}a_{jl} \frac{\partial t_{kl}}{\partial x_s} \frac{\partial x_s}{\partial x'_m}. \hfill (1.4.39)$$

Note that on the right-hand side we used the chain rule of differentiation and that there is an implied sum over the index $s$. Also note that since

$$x_s = a_{ms}x'_m \hfill (1.4.40)$$

(Problem 1.8) then

$$\frac{\partial x_s}{\partial x'_m} = a_{ms}. \hfill (1.4.41)$$

Using (1.4.41) and introducing the notation

$$\frac{\partial t'_{ij}}{\partial x'_m} \equiv t'_{ij,m}; \quad \frac{\partial t_{kl}}{\partial x_s} \equiv t_{kl,s} \hfill (1.4.42)$$

equation (1.4.39) becomes

$$t'_{ij,m} = a_{ik}a_{jl}a_{ms}t_{kl,s}. \hfill (1.4.43)$$

This shows that $t_{kl,s}$ is a third-order tensor.

Applying the same arguments to higher-order tensors shows that first-order differentiation generates a new tensor with the order increased by one. It must be emphasized, however, that in general curvilinear coordinates this differentiation does not generate a tensor.
1.4.4 The permutation symbol

This is indicated by $\epsilon_{ijk}$ and is defined by

$$
\epsilon_{ijk} = \begin{cases} 
0, & \text{if any two indices are repeated} \\
1, & \text{if } ijk \text{ is an even permutation of 123} \\
-1, & \text{if } ijk \text{ is an odd permutation of 123.}
\end{cases}
$$

(1.4.44)

A permutation is even (odd) if the number of exchanges of $i$, $j$, $k$ required to order them as 123 is even (odd). For example, to go from 213 to 123 only one exchange is needed, so the permutation is odd. On the other hand, to go from 231 to 123, two exchanges are needed: $231 \rightarrow 213 \rightarrow 123$, and the permutation is even. After considering all the possible combinations we find

$$
\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \\
\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1.
$$

(1.4.45)  (1.4.46)

The permutation symbol is also known as the alternating or Levi-Civita symbol. The definition (1.4.44) is general in the sense that it can be extended to more than three subindices. The following equivalent definition can only be used with three subindices, but is more convenient for practical uses:

$$
\epsilon_{ijk} = \begin{cases} 
0, & \text{if any two indices are repeated} \\
1, & \text{if } ijk \text{ are in cyclic order} \\
-1, & \text{if } ijk \text{ are not in cyclic order.}
\end{cases}
$$

(1.4.47)

Three different indices $ijk$ are in cyclic order if they are equal to 123 or 231 or 312. They are not in cyclic order if they are equal to 132 or 321 or 213 (see Fig. 1.4).

The permutation symbol will play an important role in the applications below, so its properties must be understood well. When using $\epsilon_{ijk}$ in equations, the values of $ijk$ are generally not specified. Moreover, it may happen that the same equation includes factors such as $\epsilon_{jik}$ and $\epsilon_{kij}$. In cases like that it is necessary to express

Fig. 1.4. Diagrams used to find out whether or not a combination of the integers 1, 2, 3, or any three indices $i$, $j$, $k$, are in cyclic order (indicated by the arrow). For example, the combination 312 is in cyclic order, while 213 is not. For arbitrary indices $i$, $j$, $k$, if the combination $ikj$ is assumed to be in cyclic order, then combinations such as $kji$ and $jik$ will also be in cyclic order, but $kij$ and $ijk$ will not.
1.4 Cartesian tensors

one of them in terms of the other. To do that assume that $jik$ is in cyclic order (Fig. 1.4) and by inspection find out whether $kij$ is in the same order or not. As it is not, $\epsilon_{kij} = -\epsilon_{jik}$.

Is the permutation symbol a tensor? Almost. If the determinant of the transformation matrix $A$ is equal to 1, then the components of $\epsilon_{ijk}$ transform according to (1.4.14a). However, if the determinant is equal to $-1$, the components of $\epsilon_{ijk}$ transform according to $-1$ times the right-hand side of (1.4.14a). Entities with this type of transformation law are known as pseudo tensors. Another well-known example is the vector product, which produces a pseudo vector, rather than a vector.

Another important aspect of $\epsilon_{ijk}$ is that its components are independent of the coordinate system (see §1.4.7). A proof of the tensorial character of $\epsilon_{ijk}$ can be found in Goodbody (1982) and McConnell (1957).

1.4.5 Applications and examples

In the following it will be assumed that the scalars, vectors, and tensors are functions of $x_1$, $x_2$, and $x_3$, as needed, and that the required derivatives exist.

(1) By contraction of the second-order tensor $u_{i,j}$ we obtain the scalar $u_{i,i}$. When this expression is written in full it is clear that it corresponds to the divergence of $u$:

$$u_{i,i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{div } u = \nabla \cdot u. \quad (1.4.48)$$

In the last term we introduced the vector operator $\nabla$ (nabla or del), which can be written as

$$\nabla = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right). \quad (1.4.49)$$

This definition of $\nabla$ is valid in Cartesian coordinate systems only.

(2) The derivative of the scalar function $f(x_1, x_2, x_3)$ with respect to $x_i$ is the $i$th component of the gradient of $f$:

$$\frac{\partial f}{\partial x_i} = (\nabla f)_i = f_{,i}. \quad (1.4.50)$$

(3) The sum of second derivatives $f_{,ii}$ of a scalar function $f$ is the Laplacian of $f$:

$$f_{,ii} = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} = \nabla^2 f. \quad (1.4.51)$$
(4) The second derivatives of the vector components $u_i$ are $u_{ij,k}$. By contraction we obtain $u_{i,jj}$, which corresponds to the $i$th component of the Laplacian of the vector $\mathbf{u}$:

$$\nabla^2 \mathbf{u} = (\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3) = (u_{1,jj}, u_{2,jj}, u_{3,jj}). \quad (1.4.52)$$

Again, this definition applies to Cartesian coordinates only. In general orthogonal coordinate systems the Laplacian of a vector is defined by

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}, \quad (1.4.53)$$

where the appropriate expressions for the gradient, divergence, and curl should be used (e.g., Morse and Feshbach, 1953; Ben-Menahem and Singh, 1981). For Cartesian coordinates (1.4.52) and (1.4.53) lead to the same expression (Problem 1.9).

(5) Show that the Kronecker delta is a second-order tensor. Let us apply the transformation law (1.4.10):

$$\delta'_{ij} = a_{il} a_{jk} \delta_{lk} = a_{il} a_{jl} = \delta_{ij}. \quad (1.4.54)$$

This shows that $\delta_{ij}$ is a tensor. Note that in the term to the right of the first equality there is a double sum over $l$ and $k$ (nine terms) but because of the definition of the delta, the only nonzero terms are those with $l = k$ (three terms), in which case delta has a value of one. The last equality comes from (1.3.19).

(6) Let $\mathbf{B}$ be the $3 \times 3$ matrix with elements $b_{ij}$. Then, the determinant of $\mathbf{B}$ is given by

$$|\mathbf{B}| \equiv \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} = \epsilon_{ijk} b_{1i} b_{2j} b_{3k} = \epsilon_{ijk} b_{i1} b_{j2} b_{k3}. \quad (1.4.55)$$

The two expressions in terms of $\epsilon_{ijk}$ correspond to the expansion of the determinant by rows and columns, respectively. It is straightforward to verify that (1.4.55) is correct. In fact, those familiar with determinants will recognize that (1.4.55) is the definition of a determinant.

(7) Vector product of $\mathbf{u}$ and $\mathbf{v}$. The $i$th component is given by

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} u_j v_k. \quad (1.4.56)$$

(8) Curl of a vector. The $i$th component is given by

$$(\text{curl } \mathbf{u})_i = (\nabla \times \mathbf{u})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k = \epsilon_{ijk} u_{k,j}. \quad (1.4.57)$$