A PRIMER OF ANALYTIC NUMBER THEORY

From Pythagoras to Riemann

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Chapter 1

Sums and Differences

I met a traveller from an antique land
Who said: Two vast and trunkless legs of stone
Stand in the desert. Near them, on the sand,
Half sunk, a shattered visage lies . . .

Percy Bysshe Shelley

1.1. Polygonal Numbers

The Greek word *gnomon* means the pointer on a sundial, and also a carpenter's square or L-shaped bar. The Pythagoreans, who invented the subject of polygonal numbers, also used the word to refer to consecutive odd integers: 1, 3, 5, 7, ... The *Oxford English Dictionary's* definition of gnomon offers the following quotation, from Thomas Stanley's *History of Philosophy* in 1687 (Stanley, 1978):

> Odd Numbers they called Gnomons, because being added to Squares, they keep the same Figures; so Gnomons do in Geometry.

In more mathematical terms, they observed that n^2 is the sum of the first n consecutive odd integers:

$$\begin{aligned} 1 &= 1^2, \\ 1+3 &= 2^2, \\ 1+3+5 &= 3^2, \\ 1+3+5+7 &= 4^2, \\ &\vdots \end{aligned}$$

Figure 1.1 shows a geometric proof of this fact; observe that each square is constructed by adding an odd number (the black dots) to the preceding square. These are the gnomons the quotation refers to.

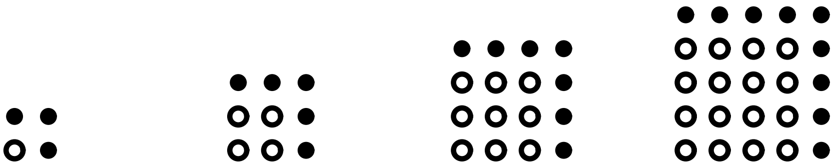

Figure 1.1. A geometric proof of the gnomon theorem.

But before we get to squares, we need to consider triangles. The TRIANGULAR NUMBERS, t_n, are the number of circles (or dots, or whatever) in a triangular array with n rows (see Figure 1.2).

Since each row has one more than the row above it, we see that

$$t_n = 1 + 2 + \cdots + n - 1 + n.$$

A more compact way of writing this, without the ellipsis, is to use the "Sigma" notation,

$$t_n = \sum_{k=1}^{n} k.$$

The Greek letter $\sum$ denotes a sum; the terms in the sum are indexed by integers between 1 and n, generically denoted k. And the thing being summed is the integer k itself (as opposed to some more complicated function of k.)

Of course, we get the same number of circles (or dots) no matter how we arrange them. In particular we can make right triangles. This leads to a clever proof of a "closed-form" expression for t_n, that is, one that does not require doing the sum. Take two copies of the triangle for t_n, one with circles and one with dots. They fit together to form a rectangle, as in Figure 1.3. Observe that the rectangle for two copies of t_n in Figure 1.3 has $n + 1$ rows and n columns, so $2t_n = n(n + 1)$, or

$$1 + 2 + \cdots + n = t_n = \frac{n(n+1)}{2}. \tag{1.1}$$

This is such a nice fact that, we will prove it two more times. The next proof is more algebraic and has a story. The story is that Gauss, as a young student, was set the task of adding together the first hundred integers by his teacher, with the hope of keeping him busy and quiet for a while. Gauss immediately came back with the answer $5050 = 100 \cdot 101/2$, because he saw the following

Figure 1.2. The triangular numbers are $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10, \ldots$.

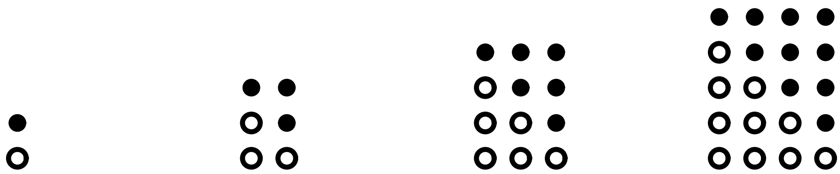

Figure 1.3. $2t_1 = 2 \cdot 1$, $2t_2 = 3 \cdot 2$, $2t_3 = 4 \cdot 3$, $2t_4 = 5 \cdot 4, \ldots$.

trick, which works for any n. Write the sum defining t_n twice, once forward and once backward:

$$\begin{array}{l} 1+\ 2+\quad \cdots + n-1 + n, \\ n + n-1 + \quad \cdots + 2 \quad + 1. \end{array}$$

Now, add vertically; each pair of terms sums to $n+1$, and there are n terms, so $2t_n = n(n+1)$ or $t_n = n(n+1)/2$.

The third proof uses mathematical induction. This is a method of proof that works when there are infinitely many theorems to prove, for example, one theorem for each integer n. The first case $n = 1$ must be proven and then it has to be shown that each case follows from the previous one. Think about a line of dominoes standing on edge. The $n = 1$ case is analogous to knocking over the first domino. The inductive step, showing that case $n-1$ implies case n, is analogous to each domino knocking over the next one in line. We will give a proof of the formula $t_n = n(n+1)/2$ by induction. The $n = 1$ case is easy. Figure 1.2 shows that $t_1 = 1$, which is equal to $(1 \cdot 2)/2$. Now we get to assume that the theorem is already done in the case of $n-1$; that is, we can assume that

$$t_{n-1} = 1 + 2 + \cdots + n - 1 = \frac{(n-1)n}{2}.$$

So

$$\begin{aligned} t_n &= 1 + 2 + \cdots + n - 1 + n = t_{n-1} + n \\ &= \frac{(n-1)n}{2} + n = \frac{(n-1)n}{2} + \frac{2n}{2} = \frac{(n+1)n}{2}. \end{aligned}$$

We have already mentioned the SQUARE NUMBERS, s_n. These are just the number of dots in a square array with n rows and n columns. This is easy; the formula is $s_n = n^2$. Nonetheless, the square numbers, s_n, are more interesting than one might think. For example, it is easy to see that the sum of two consecutive triangular numbers is a square number:

$$t_{n-1} + t_n = s_n. \tag{1.2}$$

Figure 1.4 shows a geometric proof.

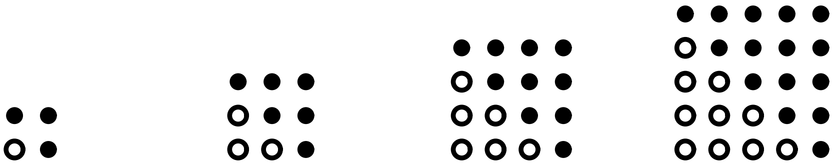

Figure 1.4. Geometric proof of Eq. (1.2).

It is also easy to give an algebraic proof of this same fact:

$$t_{n-1} + t_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{(n-1+n+1)n}{2} = n^2 = s_n.$$

Figure 1.1 seems to indicate that we can give an inductive proof of the identity

$$1 + 3 + 5 + \cdots + (2n-1) = n^2. \tag{1.3}$$

For the $n = 1$ case we just have to observe that $1 = 1^2$. And we have to show that the $n - 1$st case implies the nth case. But

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2n-3) + (2n-1) \\ = &\{1 + 3 + 5 + \cdots + (2n-3)\} + 2n - 1. \end{aligned}$$

So, by the induction hypothesis, it simplifies to

$$\begin{aligned} &(n-1)^2 + 2n - 1 \\ = &n^2 - 2n + 1 + 2n - 1 = n^2. \end{aligned}$$

Exercise 1.1.1. Since we know that $t_{n-1} + t_n = s_n$ and that $1 + 3 + \cdots + (2n-1) = s_n$, it is certainly true that

$$1 + 3 + \cdots + (2n-1) = t_{n-1} + t_n.$$

Give a *geometric* proof of this identity. That is, find a way of arranging the two triangles for t_{n-1} and t_n so that you see an array of dots in which the rows all have an odd number of dots.

Exercise 1.1.2. Give an algebraic proof of Plutarch's identity

$$8t_n + 1 = s_{2n+1}$$

using the formulas for triangular and square numbers. Now give a geometric proof of this same identity by arranging eight copies of the triangle for t_n, plus one extra dot, into a square.

Exercise 1.1.3. Which triangular numbers are also squares? That is, what conditions on m and n will guarantee that $t_n = s_m$? Show that if this happens, then we have

$$(2n + 1)^2 - 8m^2 = 1,$$

a solution to Pell's equation, which we will study in more detail in Chapter 11.

The philosophy of the Pythagoreans had an enormous influence on the development of number theory, so a brief historical diversion is in order.

Pythagoras of Samos (560–480 B.C.). Pythagoras traveled widely in Egypt and Babylonia, becoming acquainted with their mathematics. Iamblichus of Chalcis, in his *On the Pythagorean Life* (Iamblichus, 1989), wrote of Pythagoras' journey to Egypt:

> From there he visited all the sanctuaries, making detailed investigations with the utmost zeal. The priests and prophets he met responded with admiration and affection, and he learned from them most diligently all that they had to teach. He neglected no doctrine valued in his time, no man renowned for understanding, no rite honored in any region, no place where he expected to find some wonder.... He spent twenty-two years in the sacred places of Egypt, studying astronomy and geometry and being initiated ... into all the rites of the gods, until he was captured by the expedition of Cambyses and taken to Babylon. There he spent time with the Magi, to their mutual rejoicing, learning what was holy among them, acquiring perfected knowledge of the worship of the gods and reaching the heights of their mathematics and music and other disciplines. He spent twelve more years with them, and returned to Samos, aged by now about fifty-six.

(Cambyses, incidentally, was a Persian emperor who invaded and conquered Egypt in 525 B.C., ending the twenty-fifth dynasty. According to Herodotus in *The Histories*, Cambyses did many reprehensible things against Egyptian religion and customs and eventually went mad.)

The Pythagorean philosophy was that the essence of all things is numbers. Aristotle wrote in *Metaphysics* that

> [t]hey thought they found in numbers, more than in fire, earth, or water, many resemblances to things which are and become .... Since, then, all other things seemed in their whole nature to be assimilated to numbers, while numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number.

Musical harmonies, the sides of right triangles, and the orbits of different planets could all be described by ratios. This led to mystical speculations about the properties of special numbers. In astronomy the Pythagoreans had the concept of the "great year." If the ratios of the periods of the planets

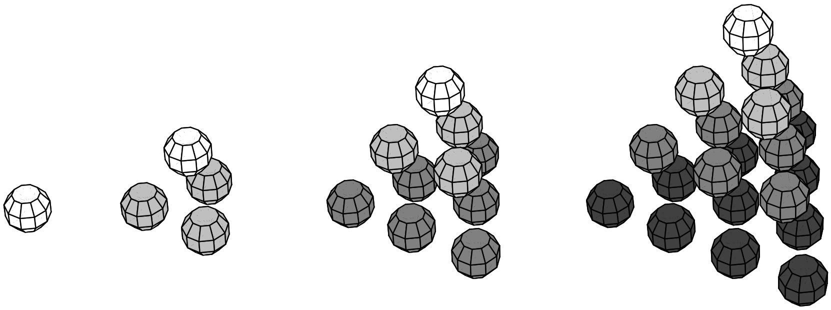

Figure 1.5. The tetrahedral numbers $T_1 = 1$, $T_2 = 4$, $T_3 = 10$, $T_4 = 20, \ldots$.

are integers, then after a certain number of years (in fact, the least common multiple of the ratios), the planets will return to exactly the same positions again. And since astrology says the positions of the planets determine events, according to Eudemus,

> ... then I shall sit here again with this pointer in my hand and tell you such strange things.

The TETRAHEDRAL NUMBERS, T_n, are three-dimensional analogs of the triangular numbers, t_n. They give the number of objects in a tetrahedral pyramid, that is, a pyramid with triangular base, as in Figure 1.5.

The kth layer of the pyramid is a triangle with t_k objects in it; so, by definition,

$$T_n = t_1 + t_2 + \cdots + t_{n-1} + t_n = \sum_{k=1}^{n} t_k. \tag{1.4}$$

Here, we use Sigma notation to indicate that the kth term in the sum is the kth triangular number, t_k.

What is the pattern in the sequence of the first few tetrahedral numbers: $1, 4, 10, 20, \ldots$? What is the formula for T_n for general n? It is possible to give a three-dimensional geometric proof that $T_n = n(n+1)(n+2)/6$. It helps to use cubes instead of spheres. First shift the cubes so they line up one above the other, as we did in two dimensions. Then try to visualize six copies of the cubes, which make up T_n filling up a box with dimensions n by $n+1$ by $n+2$. This would be a three- dimensional analog of Figure 1.3.

If this makes your head hurt, we will give another proof that is longer but not so three dimensional. In fact you can view the following explanation as a two-dimensional analog of Gauss' one dimensional proof that $t_n = n(n+1)/2$.

We will do this in the case of $n = 5$ for concreteness. From Eq. (1.4) we want to sum all the numbers in a triangle:

$$\begin{array}{c} 1 \\ 1+2 \\ 1+2+3 \\ 1+2+3+4 \\ 1+2+3+4+5 \end{array}$$

The kth row is the triangular number t_k. We take *three* copies of the triangle, each one rotated by 120°:

$$\begin{array}{ccc} 1 & 1 & 5 \\ 1+2 & 2+1 & 4+4 \\ 1+2+3 & 3+2+1 & 3+3+3 \\ 1+2+3+4 & 4+3+2+1 & 2+2+2+2 \\ 1+2+3+4+5 & 5+4+3+2+1 & 1+1+1+1+1 \end{array}$$

The rearranged triangles still have the same sum. This is the analog of Gauss taking a second copy of the sum for t_n written backward. Observe that if we add the left and center triangles together, in each row the sums are constant:

$$\begin{array}{ccccc} 1 & + & 1 & = & 2 \\ 1+2 & + & 2+1 & = & 3+3 \\ 1+2+3 & + & 3+2+1 & = & 4+4+4 \\ 1+2+3+4 & + & 4+3+2+1 & = & 5+5+5+5 \\ 1+2+3+4+5 & + & 5+4+3+2+1 & = & 6+6+6+6+6 \end{array}$$

In row k, all the entries are $k + 1$, just as Gauss found. In the third triangle, all the entries in row k are the same; they are equal to $n - k + 1$, and $k + 1$ plus $n - k + 1$ is $n + 2$.

$$\begin{array}{ccccc} 2 & + & 5 & = & 7 \\ 3+3 & + & 4+4 & = & 7+7 \\ 4+4+4 & + & 3+3+3 & = & 7+7+7 \\ 5+5+5+5 & + & 2+2+2+2 & = & 7+7+7+7 \\ 6+6+6+6+6 & + & 1+1+1+1+1 & = & 7+7+7+7+7 \end{array}$$

We get a triangle with t_n numbers in it, each of which is equal to $n + 2$. So,

$$3T_n = t_n(n + 2) = n(n + 1)(n + 2)/2,$$

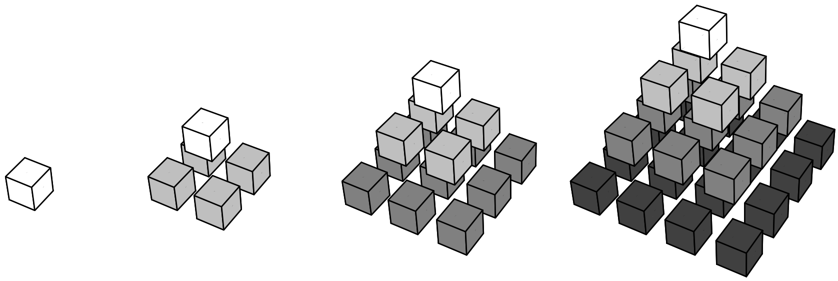

Figure 1.6. The pyramidal numbers $P_1 = 1$, $P_2 = 5$, $P_3 = 14$, $P_4 = 30, \ldots$.

and therefore,

$$T_n = n(n+1)(n+2)/6. \tag{1.5}$$

Exercise 1.1.4. Use mathematical induction to give another proof of Eq. (1.5), with T_n defined by Eq. (1.4).

The PYRAMIDAL NUMBERS, P_n, give the number of objects in a pyramid with a square base, as in Figure 1.6. The kth layer of the pyramid is a square with $s_k = k^2$ objects in it; so, by definition,

$$P_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2.$$

Since we know a relationship between square numbers and triangular numbers, we can get a formula for P_n in terms of the formula for T_n, as follows. From Eq. (1.2) we have $t_k + t_{k-1} = k^2$ for every k. This even works for $k = 1$ if we define $t_0 = 0$, which makes sense. So,

$$\begin{aligned} P_n &= \sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} \{t_k + t_{k-1}\} \\ &= \sum_{k=1}^{n} t_k + \sum_{k=1}^{n} t_{k-1} = T_n + T_{n-1}. \end{aligned}$$

According to Eq. (1.5) this is just

$$\begin{aligned} P_n &= n(n+1)(n+2)/6 + (n-1)n(n+1)/6 \\ &= n(n+1)(2n+1)/6. \end{aligned}$$

The formulas

$$1 + 2 + \cdots + n = n(n + 1)/2, \tag{1.6}$$
$$1^2 + 2^2 + \cdots + n^2 = n(n + 1)(2n + 1)/6 \tag{1.7}$$

are beautiful. Can we generalize them? Is there a formula for sums of cubes? In fact there is, due to Nicomachus of Gerasa. Nicomachus observed the interesting pattern in sums of odd numbers:

$$\begin{array}{cl} 1 & = 1^3, \\ 3 + 5 & = 2^3, \\ 7 + 9 + 11 & = 3^3, \\ 13 + 15 + 17 + 19 & = 4^3, \\ 21 + 23 + 25 + 27 + 29 & = 5^3, \\ \vdots & \quad \vdots \end{array}$$

This seems to indicate that summing consecutive cubes will be the same as summing consecutive odd numbers.

$$\begin{gathered} 1 + 3 + 5 = 1^3 + 2^3, \\ 1 + 3 + 5 + 7 + 9 + 11 = 1^3 + 2^3 + 3^3, \\ \vdots \end{gathered}$$

But how many odd numbers do we need to take? Notice that 5 is the third odd number, and $t_2 = 3$. Similarly, 11 is the sixth odd number, and $t_3 = 6$. We guess that the pattern is that the sum of the first n cubes is the sum of the first t_n odd numbers. Now Eq. (1.3) applies and this sum is just $(t_n)^2$. From Eq. (1.1) this is $(n(n + 1)/2)^2$. So it seems as if

$$1^3 + 2^3 + \cdots + n^3 = n^2(n + 1)^2/4. \tag{1.8}$$

But the preceding argument was mostly inspired guessing, so a careful proof by induction is a good idea. The base case $n = 1$ is easy because $1^3 = 1^2 \cdot 2^2/4$. Now we can assume that the $n - 1$ case

$$1^3 + 2^3 + \cdots + (n - 1)^3 = (n - 1)^2 n^2/4$$

Table 1.1. *Another proof of Nicomachus identity*

1	2	3	4	5	...
2	4	6	8	10	...
3	6	9	12	15	...
4	8	12	16	20	...
5	10	15	20	25	...
⋮	⋮	⋮	⋮	⋮	

is true and use it to prove the next case. But

$$\begin{aligned}&1^3 + 2^3 + \cdots + (n-1)^3 + n^3\\ &= \{1^3 + 2^3 + \cdots + (n-1)^3\} + n^3\\ &= \frac{(n-1)^2 n^2}{4} + n^3\end{aligned}$$

by the induction hypothesis. Now, put the two terms over the common denominator and simplify to get $n^2(n+1)^2/4$.

Exercise 1.1.5. Here's another proof that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n+1)^2/4, \tag{1.9}$$

with the details to be filled in. The entries of the multiplication table are shown in Table 1.1. Each side of the equation can be interpreted as a sum of all the entries in the table. For the left side of Eq. (1.9), form "gnomons" starting from the upper-left corner. For example, the second one is 2, 4, 2. The third one is 3, 6, 9, 6, 3, and so on.

What seems to be the pattern when you add up the terms in the kth gnomon? To prove your conjecture, consider the following questions:

1. What is the common factor of all the terms in the kth gnomon?
2. If you factor this out, can you write what remains in terms of triangular numbers?
3. Can you write what remains in terms of squares?
4. Combine these ideas to prove the conjecture you made.

The right side of Eq. (1.9) is t_n^2. Why is the sum of the n^2 entries in the first n rows and n columns equal to $t_n \cdot t_n$?

1.2. The Finite Calculus

The results in the previous sections are beautiful, but some of the proofs are almost *too* clever. In this section we will see some structure that simplifies things. This will build on skills you already have from studying calculus.

For example, if we want to go beyond triangular numbers and squares, the next step is pentagonal numbers. But the pictures are hard to draw because of the fivefold symmetry of the pentagon. Instead, consider what we've done so far:

$$\begin{array}{rccccccc} n\colon & 1 & 2 & 3 & 4 & 5 & \ldots, \\ t_n\colon & 1 & 3 & 6 & 10 & 15 & \ldots, \\ s_n\colon & 1 & 4 & 9 & 16 & 25 & \ldots. \end{array}$$

In each row, consider the differences between consecutive terms:

$$\begin{array}{rccccccc} (n+1)-n\colon & 1 & 1 & 1 & 1 & 1 & \ldots, \\ t_{n+1}-t_n\colon & 2 & 3 & 4 & 5 & 6 & \ldots, \\ s_{n+1}-s_n\colon & 3 & 5 & 7 & 9 & 11 & \ldots. \end{array}$$

There is nothing new here; in the third row, we are just seeing that each square is formed by adding an odd number (gnomon) to the previous square. If we now compute the differences again, we see

$$\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & \ldots, \\ 1 & 1 & 1 & 1 & 1 & \ldots, \\ 2 & 2 & 2 & 2 & 2 & \ldots. \end{array}$$

In each case, the second differences are constant, and the constant increases by one in each row.

For convenience we will introduce the DIFFERENCE OPERATOR, Δ, on functions $f(n)$, which gives a new function, $\Delta f(n)$, defined as $f(n+1) - f(n)$. This is an analog of derivative. We can do it again,

$$\begin{aligned} \Delta^2 f(n) &= \Delta(\Delta f)(n) \\ &= (\Delta f)(n+1) - (\Delta f)(n) \\ &= f(n+2) - 2f(n+1) + f(n), \end{aligned}$$

in an analogy with the second derivative. Think of the triangular numbers and square numbers as functions and not sequences. So,

$$\begin{aligned}
s(n) &= n^2, \\
\Delta s(n) &= (n+1)^2 - n^2 \\
&= n^2 + 2n + 1 - n^2 = 2n + 1, \\
\Delta^2 s(n) &= (2(n+1)+1) - (2n+1) = 2.
\end{aligned}$$

Based on the pattern of second differences, we expect that the pentagonal numbers, $p(n)$, should satisfy $\Delta^2 p(n) = 3$ for all n. This means that $\Delta p(n) = 3n + C$ for some constant C, since

$$\Delta(3n + C) = (3(n+1) + C) - (3n + C) = 3.$$

What about $p(n)$ itself? To correspond to the $+C$ term, we need a term, $Cn + D$ for some other constant D, since

$$\Delta(Cn + D) = (C(n+1) + D) - (Cn + D) = C.$$

We also need a term whose difference is $3n$. We already observed that for the triangular numbers, $\Delta t(n) = n + 1$. So, $\Delta t(n-1) = n$ and $\Delta(3t(n-1)) = 3n$. So,

$$p(n) = 3t(n-1) + Cn + D = 3(n-1)n/2 + Cn + D$$

for some constants C and D. We expect $p(1) = 1$ and $p(2) = 5$, because they are pentagonal numbers; so, plugging in, we get

$$\begin{aligned}
0 + C + D &= 1, \\
3 + 2C + D &= 5.
\end{aligned}$$

Solving, we get that $C = 1$ and $D = 0$, so

$$p(n) = 3(n-1)n/2 + n = n(3n-1)/2.$$

This seems to be correct, since it gives

$$\begin{array}{rcccccl}
p(n): & 1 & 5 & 12 & 22 & 35 & \ldots, \\
\Delta p(n): & 4 & 7 & 10 & 13 & 16 & \ldots, \\
\Delta^2 p(n): & 3 & 3 & 3 & 3 & 3 & \ldots.
\end{array}$$

Exercise 1.2.1. Imitate this argument to get a formula for the hexagonal numbers, $h(n)$.

The difference operator, Δ, has many similarities to the derivative d/dx in calculus. We have already used the fact that

$$\Delta(f+g)(n) = \Delta f(n) + \Delta g(n) \quad \text{and} \quad \Delta(c \cdot f)(n) = c \cdot \Delta f(n)$$

in an analogy with the corresponding rules for derivatives. But the rules are not exactly the same, since

$$\frac{d}{dx}x^2 = 2x \quad \text{but} \quad \Delta n^2 = 2n+1, \text{ not } 2n.$$

What functions play the role of powers x^m? It turns out to be the FACTORIAL POWERS

$$n^{\underline{m}} = \underbrace{n(n-1)(n-2)\cdots(n-(m-1))}_{m \text{ consecutive integers}}.$$

An empty product is 1 by convention, so

$$n^{\underline{0}} = 1, \quad n^{\underline{1}} = n, \quad n^{\underline{2}} = n(n-1), \quad n^{\underline{3}} = n(n-1)(n-2), \ldots. \qquad (1.10)$$

Observe that

$$\begin{aligned} \Delta(n^{\underline{m}}) &= (n+1)^{\underline{m}} - n^{\underline{m}} \\ &= [(n+1)\cdots(n-(m-2))] - [n\cdots(n-(m-1))]. \end{aligned}$$

The last $m-1$ factors in the first term and the first $m-1$ factors in the second term are both equal to $n^{\underline{m-1}}$. So we have

$$\begin{aligned} \Delta(n^{\underline{m}}) &= [(n+1)\cdot n^{\underline{m-1}}] - [n^{\underline{m-1}} \cdot (n-(m-1))] \\ &= \{(n+1) - (n-(m-1))\} \cdot n^{\underline{m-1}} \\ &= m \cdot n^{\underline{m-1}}. \end{aligned}$$

What about negative powers? From Eq. (1.10) we see that

$$n^{\underline{2}} = \frac{n^{\underline{3}}}{n-2}, \quad n^{\underline{1}} = \frac{n^{\underline{2}}}{n-1}, \quad n^{\underline{0}} = \frac{n^{\underline{1}}}{n-0}.$$

It makes sense to define the negative powers so that the pattern continues:

$$\begin{aligned} n^{\underline{-1}} &= \frac{n^{\underline{0}}}{n - -1} = \frac{1}{n+1}, \\ n^{\underline{-2}} &= \frac{n^{\underline{-1}}}{n - -2} = \frac{1}{(n+1)(n+2)}, \\ n^{\underline{-3}} &= \frac{n^{\underline{-2}}}{n - -3} = \frac{1}{(n+1)(n+2)(n+3)}, \\ &\vdots \end{aligned}$$

One can show that for any m, positive or negative,

$$\Delta(n^{\underline{m}}) = m \cdot n^{\underline{m-1}}. \tag{1.11}$$

Exercise 1.2.2. Verify this in the case of $m = -2$. That is, show that $\Delta(n^{\underline{-2}}) = -2 \cdot n^{\underline{-3}}$.

The factorial powers combine in a way that is a little more complicated than ordinary powers. Instead of $x^{m+k} = x^m \cdot x^k$, we have that

$$n^{\underline{m+k}} = n^{\underline{m}}(n-m)^{\underline{k}} \quad \text{for all } m, k. \tag{1.12}$$

Exercise 1.2.3. Verify this for $m = 2$ and $k = -3$. That is, show that $n^{\underline{-1}} = n^{\underline{2}}(n-2)^{\underline{-3}}$.

The difference operator, Δ, is like the derivative d/dx, and so one might ask about the operation that undoes Δ the way an antiderivative undoes a derivative. This operation is denoted Σ:

$$\Sigma f(n) = F(n), \quad \text{if } F(n) \text{ is a function with } \Delta F(n) = f(n).$$

Don't be confused by the symbol Σ; we are not computing any sums. $\Sigma f(n)$ denotes a *function*, not a number. As in calculus, there is more than one possible choice for $\Sigma f(n)$. We can add a constant C to $F(n)$, because $\Delta(C) = C - C = 0$. Just as in calculus, the rule (1.11) implies that

$$\Sigma n^{\underline{m}} = \frac{n^{\underline{m+1}}}{m+1} + C \quad \text{for } m \neq -1. \tag{1.13}$$

Exercise 1.2.4. We were already undoing the difference operator in finding pentagonal and hexagonal numbers. Generalize this to polygonal numbers with a sides, for any a. That is, find a formula for a function $f(n)$ with

$$\Delta^2 f(n) = a - 2, \quad \text{with } f(1) = 1 \text{ and } f(2) = a.$$

In calculus, the point of antiderivatives is to compute definite integrals. Geometrically, this is the area under curves. The Fundamental Theorem of Calculus says that if

$$F(x) = \int f(x)dx, \quad \text{then} \quad \int_a^b f(x)dx = F(b) - F(a).$$

We will think about this more carefully in Interlude 1, but for now the important point is the finite analog. We can use the operator Σ on functions to compute actual sums.

Theorem (Fundamental Theorem of Finite Calculus, Part I). *If*

$$\Sigma f(n) = F(n), \quad \textit{then} \quad \sum_{a \le n < b} f(n) = F(b) - F(a).$$

Proof. The hypothesis $\Sigma f(n) = F(n)$ is just another way to say that $f(n) = \Delta F(n)$. The sum on the left is

$$\begin{aligned}
\sum_{a \le n < b} f(n) &= f(a) + f(a+1) + \cdots + f(b-2) + f(b-1) \\
&= \Delta F(a) + \Delta F(a+1) + \cdots + \Delta F(b-2) + \Delta F(b-1) \\
&= (F(a+1) - F(a)) + (F(a+2) - F(a+1)) + \cdots \\
&\qquad \cdots + (F(b-1) - F(b-2)) + (F(b) - F(b-1)) \\
&= -F(a) + F(b).
\end{aligned}$$

□

Notice that it does not matter which choice of constant C we pick, because $(F(b) + C) - (F(a) + C) = F(b) - F(a)$.

As an application, we can use the fact that $\Sigma n^{\underline{1}} = \frac{n^{\underline{2}}}{2}$ to say that

$$1 + 2 + \cdots + n = \sum_{0 \le k < n+1} k^{\underline{1}} = \frac{(n+1)^{\underline{2}}}{2} - \frac{0^{\underline{2}}}{2} = \frac{n(n+1)}{2}.$$

This is formula (1.6) for triangular numbers.

Here is another example. Because

$$n^{\underline{1}} + n^{\underline{2}} = n + n(n-1) = n^2,$$

we can say that

$$\Sigma n^2 = \Sigma(n^{\underline{1}} + n^{\underline{2}}) = \frac{n^{\underline{2}}}{2} + \frac{n^{\underline{3}}}{3}.$$

So,

$$\begin{aligned}
\sum_{0 \le k < n+1} k^2 &= \left(\frac{(n+1)^{\underline{2}}}{2} + \frac{(n+1)^{\underline{3}}}{3}\right) - \left(\frac{0^{\underline{2}}}{2} + \frac{0^{\underline{3}}}{3}\right) \\
&= \frac{(n+1)n}{2} + \frac{(n+1)n(n-1)}{3} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

This is just Eq. (1.7) again.

Exercise 1.2.5. First, verify that

$$n^{\underline{1}} + 3n^{\underline{2}} + n^{\underline{3}} = n^3.$$

Now use this fact to find formulas for

$$\sum_{0 \le k < n+1} k^3.$$

Your answer should agree with formula (1.8).

In fact, one can do this for any exponent m. We will see that there are integers called STIRLING NUMBERS, $\left\{ {m \atop k} \right\}$, which allow you to write ordinary powers in terms of factorial powers:

$$n^m = \sum_{k=0}^{m} \left\{ {m \atop k} \right\} n^{\underline{k}}. \tag{1.14}$$

In the preceding example, we saw that

$$\left\{ {2 \atop 0} \right\} = 0, \left\{ {2 \atop 1} \right\} = 1, \left\{ {2 \atop 2} \right\} = 1.$$

In the first part of Exercise 1.2.5, you verified that

$$\left\{ {3 \atop 0} \right\} = 0, \left\{ {3 \atop 1} \right\} = 1, \left\{ {3 \atop 2} \right\} = 3, \left\{ {3 \atop 3} \right\} = 1.$$

Exercise 1.2.6. Use the Stirling numbers

$$\left\{ {4 \atop 0} \right\} = 0, \left\{ {4 \atop 1} \right\} = 1, \left\{ {4 \atop 2} \right\} = 7, \left\{ {4 \atop 3} \right\} = 6, \left\{ {4 \atop 4} \right\} = 1$$

to show that

$$1^4 + 2^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30. \tag{1.15}$$

The Stirling numbers are sort of like the binomial coefficients $\binom{m}{k}$. Binomial coefficients are found in Pascal's triangle, which you have probably seen:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The first and last entry in each row is always 1; the rest are computed by adding the two binomial coefficients on either side in the previous row. Suppose we make a similar triangle for the Stirling numbers. The Stirling number $\left\{ {m \atop k} \right\}$ is the kth entry in row m here:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 7 & & 6 & & 1 & \\ 1 & & 15 & & 25 & & 10 & & 1 \end{array}$$

Exercise 1.2.7. Try to find the pattern in this triangle, similar to Pascal's. Here's a hint, but don't read it unless you're really stuck. The 3 is computed from the 1 and the second entry, also a 1, above it. The 7 is computed from the 1 and the second entry, a 3, above it. The 6 is computed from the 3 and the third entry, a 1, above it. What is the pattern?

Fill in the next row of Stirling's triangle.

In fact, if we make this a little more precise, we can prove the theorem now. First, though, we need to define

$$\left\{ {m \atop 0} \right\} = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m > 0, \end{cases} \qquad \text{and } \left\{ {m \atop k} \right\} = 0, \qquad \text{if } k > m \text{ or } k < 0.$$

Theorem. *If we now* define *the Stirling numbers by the recursion you discovered, that is,*

$$\left\{ {m \atop k} \right\} = k \left\{ {m-1 \atop k} \right\} + \left\{ {m-1 \atop k-1} \right\},$$

then Eq. (1.14) is true.

Notice that we have switched our point of view; the recursion is now the definition and the property (1.14) that we are interested in is a theorem. This is perfectly legal, as long as we make it clear that is what is happening. You may have indexed things slightly differently; make sure your recursion is equivalent to this one.

Proof. We can prove Eq. (1.14) by induction. The case of $m = 1$ is already done. From the boundary conditions ($k > m$ or $k < 0$) defined earlier, we can