Introduction to Classical Integrable Systems

OLIVIER BABELON
Laboratoire de Physique Théorique et Hautes Energies, Universités Paris VI–VII

DENIS BERNARD
Service de Physique Théorique de Saclay, Gif-sur-Yvette

MICHEL TALON
Laboratoire de Physique Théorique et Hautes Energies, Universités Paris VI–VII

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

© O. Babelon, D. Bernard & M. Talon 2003

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 2003

A catalogue record for this book is available from the British Library

Library of Congress Cataloguing in Publication data

Babelon, Olivier, 1951–
Introduction to classical integrable systems / Olivier Babelon, Denis Bernard, Michel Talon.
p. cm. – (Cambridge monographs on mathematical physics)
Includes bibliographical references and index.
ISBN 0 521 82267 X
1. Dynamics. 2. Hamiltonian systems. I. Bernard, Denis, 1961–
II. Talon, Michel, 1952– III. Title. IV. Series.
QA845 .B32 2003
531′.163–dc21 2002034955

ISBN 0 521 82267 X hardback

Contents

1	**Introduction**	**1**
2	**Integrable dynamical systems**	**5**
2.1	Introduction	5
2.2	The Liouville theorem	7
2.3	Action–angle variables	10
2.4	Lax pairs	11
2.5	Existence of an r-matrix	13
2.6	Commuting flows	17
2.7	The Kepler problem	17
2.8	The Euler top	19
2.9	The Lagrange top	20
2.10	The Kowalevski top	22
2.11	The Neumann model	23
2.12	Geodesics on an ellipsoid	25
2.13	Separation of variables in the Neumann model	27
3	**Synopsis of integrable systems**	**32**
3.1	Examples of Lax pairs with spectral parameter	33
3.2	The Zakharov–Shabat construction	35
3.3	Coadjoint orbits and Hamiltonian formalism	41
3.4	Elementary flows and wave function	49
3.5	Factorization problem	54
3.6	Tau-functions	59
3.7	Integrable field theories and monodromy matrix	62
3.8	Abelianization	65
3.9	Poisson brackets of the monodromy matrix	72
3.10	The group of dressing transformations	74

3.11 Soliton solutions 79

4 Algebraic methods 86
4.1 The classical and modified Yang–Baxter equations 86
4.2 Algebraic meaning of the classical Yang–Baxter equations 89
4.3 Adler–Kostant–Symes scheme 92
4.4 Construction of integrable systems 94
4.5 Solving by factorization 96
4.6 The open Toda chain 97
4.7 The r-matrix of the Toda models 100
4.8 Solution of the open Toda chain 105
4.9 Toda system and Hamiltonian reduction 109
4.10 The Lax pair of the Kowalevski top 115

5 Analytical methods 124
5.1 The spectral curve 125
5.2 The eigenvector bundle 130
5.3 The adjoint linear system 138
5.4 Time evolution 142
5.5 Theta-functions formulae 145
5.6 Baker–Akhiezer functions 149
5.7 Linearization and the factorization problem 153
5.8 Tau-functions 154
5.9 Symplectic form 156
5.10 Separation of variables and the spectral curve 162
5.11 Action–angle variables 164
5.12 Riemann surfaces and integrability 167
5.13 The Kowalevski top 169
5.14 Infinite-dimensional systems 175

6 The closed Toda chain 178
6.1 The model 178
6.2 The spectral curve 181
6.3 The eigenvectors 182
6.4 Reconstruction formula 184
6.5 Symplectic structure 191
6.6 The Sklyanin approach 193
6.7 The Poisson brackets 196
6.8 Reality conditions 200

7 The Calogero–Moser model 206
7.1 The spin Calogero–Moser model 206

7.2 Lax pair 208
7.3 The r-matrix 210
7.4 The scalar Calogero–Moser model 214
7.5 The spectral curve 216
7.6 The eigenvector bundle 218
7.7 Time evolution 220
7.8 Reconstruction formulae 221
7.9 Symplectic structure 223
7.10 Poles systems and double-Bloch condition 226
7.11 Hitchin systems 232
7.12 Examples of Hitchin systems 239
7.13 The trigonometric Calogero–Moser model 244

8 Isomonodromic deformations 249
8.1 Introduction 249
8.2 Monodromy data 251
8.3 Isomonodromy and the Riemann–Hilbert problem 262
8.4 Isomonodromic deformations 264
8.5 Schlesinger transformations 270
8.6 Tau-functions 272
8.7 Ricatti equation 277
8.8 Sato's formula 278
8.9 The Hirota equations 280
8.10 Tau-functions and theta-functions 282
8.11 The Painlevé equations 290

9 Grassmannian and integrable hierarchies 299
9.1 Introduction 299
9.2 Fermions and $GL(\infty)$ 303
9.3 Boson–fermion correspondence 308
9.4 Tau-functions and Hirota bilinear identities 311
9.5 The KP hierarchy and its soliton solutions 314
9.6 Fermions and Grassmannians 316
9.7 Schur polynomials 322
9.8 From fermions to pseudo-differential operators 328
9.9 The Segal–Wilson approach 331

10 The KP hierarchy 338
10.1 The algebra of pseudo-differential operators 338
10.2 The KP hierarchy 341
10.3 The Baker–Akhiezer function of KP 344
10.4 Algebro-geometric solutions of KP 348

10.5 The tau-function of KP 352
10.6 The generalized KdV equations 355
10.7 KdV Hamiltonian structures 359
10.8 Bihamiltonian structure 363
10.9 The Drinfeld–Sokolov reduction 364
10.10 Whitham equations 370
10.11 Solution of the Whitham equations 379

11 The KdV hierarchy 382
11.1 The KdV equation 382
11.2 The KdV hierarchy 386
11.3 Hamiltonian structures and Virasoro algebra 392
11.4 Soliton solutions 394
11.5 Algebro-geometric solutions 398
11.6 Finite-zone solutions 408
11.7 Action-angle variables 414
11.8 Analytical description of solitons 419
11.9 Local fields 425
11.10 Whitham's equations 433

12 The Toda field theories 443
12.1 The Liouville equation 443
12.2 The Toda systems and their zero-curvature representations 445
12.3 Solution of the Toda field equations 447
12.4 Hamiltonian formalism 454
12.5 Conformal structure 456
12.6 Dressing transformations 463
12.7 The affine sinh-Gordon model 467
12.8 Dressing transformations and soliton solutions 471
12.9 N-soliton dynamics 474
12.10 Finite-zone solutions 481

13 Classical inverse scattering method 486
13.1 The sine-Gordon equation 486
13.2 The Jost solutions 487
13.3 Inverse scattering as a Riemann–Hilbert problem 496
13.4 Time evolution of the scattering data 497
13.5 The Gelfand–Levitan–Marchenko equation 498
13.6 Soliton solutions 502
13.7 Poisson brackets of the scattering data 505
13.8 Action–angle variables 510

14 Symplectic geometry **516**
14.1 Poisson manifolds and symplectic manifolds 516
14.2 Coadjoint orbits 522
14.3 Symmetries and Hamiltonian reduction 525
14.4 The case $M = T^*G$ 532
14.5 Poisson–Lie groups 534
14.6 Action of a Poisson–Lie group on a symplectic manifold 538
14.7 The groups G and G^* 540
14.8 The group of dressing transformations 542

15 Riemann surfaces **545**
15.1 Smooth algebraic curves 545
15.2 Hyperelliptic curves 547
15.3 The Riemann–Hurwitz formula 549
15.4 The field of meromorphic functions of a Riemann surface 549
15.5 Line bundles on a Riemann surface 551
15.6 Divisors 553
15.7 Chern class 554
15.8 Serre duality 554
15.9 The Riemann–Roch theorem 556
15.10 Abelian differentials 559
15.11 Riemann bilinear identities 560
15.12 Jacobi variety 562
15.13 Theta-functions 563
15.14 The genus 1 case 567
15.15 The Riemann–Hilbert factorization problem 568

16 Lie algebras **571**
16.1 Lie groups and Lie algebras 571
16.2 Semi-simple Lie algebras 574
16.3 Linear representations 580
16.4 Real Lie algebras 583
16.5 Affine Kac–Moody algebras 587
16.6 Vertex operator representations 592

Index 599

1
Introduction

The aim of this book is to introduce the reader to classical integrable systems. Because the subject has been developed by several schools having different perspectives, it may appear fragmented at first sight. We develop here the thesis that it has a profound unity and that the various approaches are simply changes of point of view on the same underlying reality. The more one understands each approach, the more one sees their unity. At the end one gets a very small set of interconnected methods.

This fundamental fact sets the tone of the book. We hope in this way to convey to the reader the extraordinary beauty of the structures emerging in this field, which have illuminated many other branches of theoretical physics.

The field of integrable systems is born together with Classical Mechanics, with a quest for exact solutions to Newton's equations of motion. It turned out that apart from the Kepler problem which was solved by Newton himself, after two centuries of hard investigations, only a handful of other cases were found. In the nineteenth century, Liouville finally provided a general framework characterizing the cases where the equations of motion are "solvable by quadratures". All examples previously found indeed pertained to this setting. The subject stayed dormant until the second half of the twentieth century when Gardner, Greene, Kruskal and Miura invented the Classical Inverse Scattering Method for the Korteweg–de Vries equation, which had been introduced in fluid mechanics. Soon afterwards, the Lax formulation was discovered, and the connection with integrability was unveiled by Faddeev, Zakharov and Gardner. This was the signal for a revival of the domain leading to an enormous amount of results, and truly general structures emerged which organized the subject. More recently, the extension of these results to Quantum Mechanics already led to remarkable results and is still a very active field of research.

Let us give a general overview of the ideas we present in this book. They all find their roots in the notion of Lax pairs. It consists of presenting the equations of motion of the system in the form $\dot{L}(\lambda) = [M(\lambda), L(\lambda)]$, where the matrices $L(\lambda)$ and $M(\lambda)$ depend on the dynamical variables and on a parameter λ called the spectral parameter, and $[\ ,\ ]$ denotes the commutator of matrices. The importance of Lax pairs stems from the following simple remark: the Lax equation is an isospectral evolution equation for the Lax matrix $L(\lambda)$. It follows that the curve defined by the equation $\det(L(\lambda) - \mu I) = 0$ is time-independent. This curve, called the spectral curve, can be seen as a Riemann surface. Its moduli contain the conserved quantities. This immediately introduces the two main structures into the theory: groups enter through the Lie algebra involved in the commutator $[M, L]$, while complex analysis enters through the spectral curve.

As integrable systems are rather rare, one naturally expects strong constraints on the matrices $L(\lambda)$ and $M(\lambda)$. Constructing consistent Lax matrices may be achieved by appealing to factorization problems in appropriate groups. Taking into account the spectral parameter promotes this group to a loop group. The factorization problem may then be viewed as a Riemann–Hilbert problem, a central tool of this subject.

In the group theoretical setting, solving the equations of motion amounts to solving the factorization problem. In the analytical setting, solutions are obtained by considering the eigenvectors of the Lax matrix. At any point of the spectral curve there exists an eigenvector of $L(\lambda)$ with eigenvalue μ. This defines an analytic line bundle $\mathcal{L}$ on the spectral curve with prescribed Chern class. The time evolution is described as follows: if $\mathcal{L}(t)$ is the line bundle at time t then $\mathcal{L}(t)\mathcal{L}^{-1}(0)$ is of Chern class 0, i.e. is a point on the Jacobian of the spectral curve. It is a beautiful result that this point evolves linearly on the Jacobian. As a consequence, one can express the dynamical variables in terms of theta-functions defined on the Jacobian of the spectral curve. The two methods are related as follows: the factorization problem in the loop group defines transition functions for the line bundle $\mathcal{L}$.

The framework can be generalized by replacing the Lax matrix by the first order differential equation $(\partial_\lambda - M_\lambda(\lambda))\Psi = 0$, where $M_\lambda(\lambda)$ depends rationally on λ. The solution Ψ acquires non-trivial monodromy when λ describes a loop around a pole of M_λ. The isomonodromy problem consists of finding all M_λ with prescribed monodromy data. The solutions depend, in general, on a number of continuous parameters. The deformation equations with respect to these parameters form an integrable system. The theta-functions of the isospectral approach are then promoted to more general objects called the tau-functions.

One can study the behaviour around each singularity of the differential operator quite independently. In the group theoretical version, the above extension of the framework corresponds to centrally extending the loop groups. Around a singularity the most general extended group is the group $GL(\infty)$ which corresponds to the KP hierarchy. It can be represented in a fermionic Fock space. Fermionic monomials acting on the vacuum yield decomposed vectors, which describe an infinite Grassmannian introduced by Sato. In this setting, the time flows are induced by the action of commuting one-parameter subgroups, and the tau-function is defined on the Grassmannian, i.e. the orbit of the vacuum, and characterizes it. Finally the Plücker equations of the Grassmannian are identified with the equations of motion, written in the bilinear Hirota form.

We have tried, as much as possible, to make the book self-contained, and to achieve that each chapter can be studied quite independently. Generally, we first explain methods and then show how they can be applied to particular examples, even though this does not correspond to the historical development of the subject.

In Chapter 2 we introduce the classical definition of integrable systems through the Liouville theorem. We present the Lax pair formulation, and describe the symplectic structure which is encoded into the so-called r-matrix form. In Chapter 3 we explain how to construct Lax pairs with spectral parameter, for finite and infinite-dimensional systems. The Lax matrix may be viewed as an element of a coadjoint orbit of a loop group. This introduces immediately a natural symplectic structure and a factorization problem in the loop group. We also introduce, at this early stage, the notion of tau-functions. In Chapter 4 we discuss the abstract group theoretical formulation of the theory. We then describe the analytical aspects of the theory in Chapter 5. In this setting, the action variables are g moduli of the spectral curve, a Riemann surface of genus g, and the angle variables are g points on it. We illustrate the general constructions by the examples of the closed Toda chain in Chapter 6, and the Calogero model in Chapter 7.

The following two Chapters, 8 and 9, describe respectively the isomonodromic deformation problem and the infinite Grassmannian. Soliton solutions are obtained using vertex operators. Chapters 10 and 11 are devoted to the classical study of the KP and KdV hierarchies. We develop and use the formalism of pseudo-differential operators which allows us to give simple proofs of the main formal properties. Finite-zone solutions of KdV allow us to make contact with integrable systems of finite dimensionality and soliton solutions.

In the next Chapter, 12, we study the class of Toda and sine-Gordon field theories. We use this opportunity to exhibit the relations between

their conformal and integrable properties. The sine-Gordon model is presented in the framework of the Classical Inverse Scattering Method in Chapter 13. This very ingenious method is exploited to solve the sine-Gordon equation.

The last three chapters may be viewed as mathematical appendices, provided to help the reader. First we present the basic facts of symplectic geometry, which is the natural language to speak about Classical Mechanics and integrable systems. Since mathematical tools from Riemann surfaces and Lie groups are used almost everywhere, we have written two chapters presenting them in a concise way. We hope that they will be useful at least as an introduction and to fix notations.

Let us say briefly how we have limited our discussion. First we choose to remain consistently at a relatively elementary mathematical level, and have been obliged to exclude some important developments which require more advanced mathematics. We put the emphasis on methods and we have not tried to make an exhaustive list of integrable systems. Another aspect of the theory we have touched only very briefly, through the Whitham equations, is the study of perturbations of integrable systems. All these subjects are very interesting by themselves, but the present book is big enough!

A most active field of recent research is concerned with quantum integrable systems or the closely related field of exactly soluble models in statistical mechanics. When writing this book we always had the quantum theory present in mind, and have introduced all classical objects which have a well-known quantum counterpart, or are semi-classical limits of quantum objects. This explains our emphasis on Hamiltonians methods, Poisson brackets, classical r-matrices, Lie–Poisson properties of dressing transformations and the method of separation of variables. Although there is nothing quantum in this book, a large part of the apparatus necessary to understand the literature on quantum integrable systems is in fact present.

The bibliography for integrable systems would fill a book by itself. We have made no attempt to provide one. Instead, we give, at the end of each chapter, a short list of references, which complements and enhances the material presented in the chapter, and we highly encourage the reader to consult them. Of course these references are far from complete, and we apologize to the numerous authors having contributed to the domain, and whose due credit is not acknowledged. Finally we want to thank our many colleagues from whom we learned so much and with whom we have discussed many parts of this book.