

*The Cambridge Companion to*  
**BERTRAND  
RUSSELL**

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Edited by

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## LIST OF ABBREVIATIONS USED IN CITATIONS

In this book, like many others on Russell, abbreviations have been used to identify his most frequently cited works. The list below identifies not only the work but also the edition cited in this volume (in the case of books, generally the first British edition). In the case of *The Problems of Philosophy*, however, there are a number of printings with different paginations, and references here are given to both the first British edition and to a widely available reprint, the pagination of which is shared by a number of other reprints. *Principia Mathematica* poses different problems: a new introduction and several new appendices, representing a different philosophical point of view, were added for the second edition of 1925–7. These major changes did not affect the pagination of the original. Nonetheless, pagination was altered as a result of the first two volumes being reset. The first edition is extremely rare and the second is, in any case, preferable since the resetting allowed misprints to be corrected. Accordingly, whenever *Principia* is cited, the reference is to the second edition; but when material is referred to which is only to be found in the second edition, the citation is to 'PM2' rather than to 'PM'.

The use of acronyms is much more selective in the case of Russell's articles. Wherever possible, the definitive version of the text as established in *The Collected Papers of Bertrand Russell* is cited. Some contributors to the volume cited other widely used editions. In such cases, the original citations have been kept and citations to the *Collected Papers* added. The volumes of the *Collected Papers* cited in this book are as follows:

*Papers 1: Cambridge Essays: 1888–99*. Edited by Kenneth Blackwell, Andrew Brink, Nicholas Griffin, Richard A. Rempel, and John G. Slater. London: Allen and Unwin, 1983.

*Papers 2: Philosophical Papers: 1896–99.* Edited by Nicholas Griffin and Albert C. Lewis. London: Unwin Hyman, 1990.

*Papers 3: Towards the 'Principles of Mathematics', 1900–2.* Edited by Gregory H. Moore. London: Routledge, 1993.

*Papers 4: Foundations of Logic, 1903–5.* Edited by Alasdair Urquhart. London: Routledge, 1994.

*Papers 6: Logical and Philosophical Papers, 1909–13.* Edited by John G. Slater. London: Routledge, 1992.

*Papers 7: Theory of Knowledge: The 1913 Manuscript.* Edited by Elizabeth Ramsden Eames and Kenneth Blackwell. London: Allen and Unwin, 1984.

*Papers 8: The Philosophy of Logical Atomism and Other Essays: 1914–19.* Edited by John G. Slater. London: Allen and Unwin, 1986.

*Papers 9: Essays on Language, Mind, and Matter: 1919–26.* Edited by John G. Slater. London: Unwin Hyman, 1988.

*Papers 10: A Fresh Look at Empiricism: 1927–42.* Edited by John G. Slater. London: Routledge, 1996.

*Papers 11: Last Philosophical Testament: 1943–68.* Edited by John G. Slater. London: Routledge, 1997.

*Papers 28: Man's Peril 1954–5.* Edited by Andrew G. Bone. London: Routledge, 2003.

Other works by Russell are cited as follows:

*Auto* *The Autobiography of Bertrand Russell* 3 vols. (London: Allen and Unwin, 1967–9).

*AMa* *The Analysis of Matter* (London: Kegan Paul, 1927).

*AMi* *The Analysis of Mind* (London: Allen and Unwin, 1921).

*AMR* 'An Analysis of Mathematical Reasoning' (1898), in *Papers 2*, pp. 154–242.

*EA* *Essays in Analysis*. Edited by Douglas Lackey (London: Allen and Unwin, 1974).

*EFG* *An Essay on the Foundations of Geometry* (Cambridge: Cambridge University Press, 1897; New York: Dover, 1956).

*HK* *Human Knowledge: Its Scope and Limits* (London: Allen and Unwin, 1984).

*HWP* *History of Western Philosophy* (London: Allen and Unwin, 1946).

- HSEP* *Human Society in Ethics and Politics* (London: Allen and Unwin, 1954).
- IMP* *Introduction to Mathematical Philosophy* (London: Allen and Unwin, 1919).
- IMT* *Inquiry into Meaning and Truth* (London: Allen and Unwin, 1940).
- KAKD* 'Knowledge by Acquaintance and Knowledge by Description' (1911), in *Papers* 6, pp. 147–61.
- LA* 'Logical Atomism' (1924), in *Papers* 9, pp. 160–79; and *LK*, pp. 323–43.
- LK* *Logic and Knowledge*. Edited by R.C. Marsh (London: Allen and Unwin, 1956).
- ML* *Mysticism and Logic and Other Essays* (London: Longmans Green, 1918).
- MLT* 'Mathematical Logic as based on the Theory of Types', *American Journal of Mathematics*, 30 (1908), pp. 222–62; reprinted in *LK*, pp. 59–102.
- MPD* *My Philosophical Development* (London: Allen and Unwin, 1959).
- MTCA* 'Meinong's Theory of Complexes and Assumptions' (1904), in *Papers* 4, pp. 431–74; and *EA*, pp. 21–76.
- OD* 'On Denoting' (1905), in *Papers* 4, pp. 414–27; *LK*, pp. 41–56; *EA*, pp. 103–19.
- OI* 'On "Insolubilia" and their Solution by Symbolic Logic' in *EA*, pp. 190–214. (Originally published under the title 'Les Paradoxes de Logique' in the *Revue de métaphysique et de morale* 14 (1906), pp. 627–50.)
- OKEW* *Our Knowledge of the External World* (London: Allen and Unwin, 1926; first edition, 1914).
- OOP* *Outline of Philosophy* (London: Allen and Unwin, 1927).
- OP* 'On Propositions: What They Are and How They Mean' (1919), in *Papers* 8, pp. 276–306; *LK*, pp. 285–320.
- PLA* 'The Philosophy of Logical Atomism' (1918), in *Papers* 8, pp. 157–244; *LK*, pp. 177–281.
- PM* *Principia Mathematica* (with A.N. Whitehead), 3 vols. (Cambridge: Cambridge University Press, 1925–7; first edition, 1910–13).
- PM2* 'Introduction to Second Edition' in *PM* vol. 1, pp. i–xlvi and Appendices A, B, C, in *PM* vol. 1, pp. 635–66.

- POM *The Principles of Mathematics* (London: Allen and Unwin, 1964; first edition, 1903).
- POP *The Problems of Philosophy* (London: Williams and Norgate, 1912).
- POP<sub>2</sub> *The Problems of Philosophy* (Oxford: Oxford University Press, 1959).
- PSR *Principles of Social Reconstruction* (London: Allen and Unwin, 1916).
- RMDP 'The Regressive Method of Discovering the Premises of Mathematics,' in *EA*, pp. 272–83.
- ROE *Russell on Ethics*. Edited by C. Pigden (London: Routledge, 1999).
- RS *Religion and Science* (London: Thornton Butterworth, 1935).
- RSDP 'The Relation of Sense-Data to Physics' (1914), in *Papers* 8, pp. 3–26.
- RTC 'Reply to Criticisms' (1944), in *Papers* 11, pp. 18–66.
- SMP 'On Scientific Method in Philosophy' in *Papers* 8, pp. 55–73.
- TK *Theory of Knowledge. The 1913 Manuscript = Papers* 7.

# 1 Mathematics in and behind Russell's Logicism, and Its Reception

Most of the interest in Russell's work in logic has lain in its philosophical consequences; however, the main thrust came from a mathematical aim, which is the concern of this article. Russell took over a logic of propositional and predicate calculi, added to it a logic of relations (predicates of more than one variable), and thought that "all" mathematics could be delivered from such resources, not merely the methods of reasoning required but also the objects. What is the prehistory of this 'logicism', as it has become known?<sup>1</sup> How much mathematics was captured by it? Which techniques were used to effect the construction? How was it received? Figure 1 gives a flow chart of the story.

## I. THE FOUNDATIONS OF MATHEMATICAL ANALYSIS: CANTOR AND PEANO

The parent branch of mathematics was mathematical analysis, created by A.L. Cauchy (1789–1857) from the 1820s. The theory of limits was the underlying doctrine, upon which were constructed the theory of functions, the convergence of infinite series, and the differential and integral calculus. A main feature was to display proofs in full detail. The presence of logic was also raised, in that he considered

<sup>1</sup> Russell gave his position no particular name, but 'logistic' was used from 1904 to refer both to it and to the different one (explained below) held by Peano and his followers. 'Logicism' is due to Carnap 1929, 2–3, a book noted in §10; it also appeared, perhaps independently, in Fraenkel 1928 (title of the section on p. 244, explanation on p. 263). The word had taken a different meaning earlier, especially with Wilhelm Wundt, in the general context of phenomenology.



systematically the necessary and/or sufficient conditions for the truth of theorems; however, he did not treat logic explicitly.<sup>2</sup>

Gradually, this approach gained favour among those concerned with rigour in the subject, especially Karl Weierstrass (1815–1897) with his teaching from the late 1850s at Berlin University. He and his many followers prosecuted the same methodology and added further refinements to Cauchy's basic definitions (Dugac 1973). Another main imperative was to reduce the indefinables in the subject to the cardinal numbers, by introducing definitions of rational and especially irrational numbers: Weierstrass proffered one, but the best known theory was the 'cut' method (1872) of Richard Dedekind (1831–1916), in which real numbers were divided by a (arbitrary) cut  $C$  through their continuum at value  $V$ , and if there was neither a maximum rational value less than  $V$  nor a minimum one greater than  $V$ , then  $C$  was taken to define  $V$  as an irrational number.

Two further features formed the major influences upon Russell. One was the development of set theory by Georg Cantor (1845–1918) from the early 1870s, soon after he graduated from Berlin University. (To conform to Russell's usage, I shall speak of 'class' rather than 'set'; however, 'set theory' is now too durable to alter.) Inspired by a technical problem in mathematical analysis, Cantor offered a definition of irrational numbers and also developed the topology of classes of points. Distinguishing membership of an element to a class from the (im)proper inclusion of sub-classes, he worked out from the notion of the limit point of members of a class and the 'derived' set of such limit points and then considered its own derived class, and so on – transfinitely indeed, for it was in considering the infinitieth derived class and its own derived class(es) that he stumbled into the actual infinite in the first place. Then he defined various kinds of class in terms of relationships to its derived classes (closed, dense-itself, perfect, and so on). He secured the interest of Dedekind, who contributed some details.

Over the years, broader ambitions for mathematics emerged, with which Russell was to be more concerned. Cantor published details on the following features, especially in a long paper in two parts of 1895 and 1897 in *Mathematische Annalen* (Dauben 1979, esp.

<sup>2</sup> There are various studies of these developments; see especially Bottazzini 1986, Grattan-Guinness 1980, chs. 3–5; and 2000, chs. 2–5. On Cauchy's phase and its own background, see also Grattan-Guinness 1990, esp. chs. 3–4, 10–11.

Chapters 8–11):

- 1) the theory of transfinitely large ordinals and cardinals, which revolutionised understanding of the actual infinite;
- 2) different types of order of classes; the 'well order' of the positive cardinals was the premier form, but other important kinds included those of the negative cardinals, rational numbers, and real numbers, with the latter bearing upon the notion of continuity;
- 3) set theory as the basis of mathematics, starting out from the process of taking any class and abstracting from it the nature of its members to leave its 'order-type' and abstracting that to lay bare its cardinal;
- 4) methods of forming classes from given ones, especially 'covering', where the class of all sub-classes of any class  $S$  was formed and shown to have a cardinality greater than that of  $S$ .

The second main influence upon Russell came from Giuseppe Peano (1858–1932). Although not a student of Weierstrass, he was much impressed by the aspirations for rigour: one of his first publications was a collection of notes to a textbook on mathematical analysis of 1884 by his former teacher Angelo Genocchi, where he exposed various pertinent subtleties (counter-examples to apparently true theorems, and so on). By the end of the decade he was applying the method of axiomatisation to various branches of mathematics. He started in 1888 with the algebraic methods of the German mathematician Hermann Grassmann, in effect axiomatising the notion of a finite vector space. Then he switched next year to arithmetic, where he reduced the integers to three indefinables: initial ordinal, the successor operation, and proof by mathematical induction. (The year before Dedekind had offered a similar version, with a deeper understanding of induction.) Peano also soon treated geometry, where he found some of the axioms that Euclid had taken for granted; and in 1890 he tackled in *Mathematische Annalen* a problem concerning differential equations by means which used symbols as much as possible and reduced words to a remarkable minimum.

This procedure was to become Peano's principal contribution to raising the level of rigour in mathematics. Aware of the fine distinctions made by the Weierstrassians, he decided that ordinary language could be fatally ambiguous in such contexts; so he symbolised not

only the mathematics involved but also the 'mathematical logic' (his name in this sense) and the attendant set theory, which he associated with predicates, or 'propositional functions'.

In the 1890s Peano formed a 'society of mathematicians' to help him develop this programme (Cassina 1961a, 1961b). A journal began in 1891; initially called *Rivista di matematica*, it was also known as *Revue des mathématiques* from its fifth volume (1895) and ended three volumes later in 1906. In 1895 he also began to edit a primer of logico-mathematical theories, called *Formulaire de mathématiques* in the first edition and continued in further and larger editions of 1897–9, 1901, 1902–3, and finally 1905–8. Dozens of colleagues and students contributed to these projects over the years; the three most prominent 'Peanists' (as they became known) were Cesari Burali-Forti (1861–1931), Mario Pieri (1860–1913), and Alessandro Padoa (1868–1937), who also published various papers of their own in other journals (Rodriguez-Consuegra 1991, Chapter 3). Burali-Forti also put out the first textbook on *Logica matematica* in 1894.

## II. RUSSELL'S WAY INTO THE FOUNDATIONS OF MATHEMATICS

Also in 1894 Russell was studying philosophy at Cambridge having recently taken Part 1 of the mathematical Tripos. While the latter occupied a prominent place in the university, it was being roundly criticised as providing only a bunch of skills; for example, none of the developments described above were handled. Thus, young Russell's reaction in switching to philosophy for his Part 2 is symptomatic.

After graduation, Russell merged these two trainings in a search for a foundation of mathematics, starting with a Trinity College Fellowship dissertation in 1895, which he revised into the book *An Essay on the Foundations of Geometry* (1897). The philosophy brought to bear was the neo-Hegelian tradition, then dominant at Cambridge, which he used to combat empiricism. Dividing geometry into its 'projective' and 'metrical' parts by the criterion that the former involved only order but the latter also 'introduces the new idea of motion' in order to effect measurement, he construed each geometry as a construction made by us given space and time as an 'externality'. While exercised with skill, this philosophical basis did not yield results satisfactory for mathematics. It also brought him a rather

bruising first contact with Henri Poincaré in the *Revue de métaphysique et de morale*; however, he also gained the support and friendship of Louis Couturat, who helped him over the editing of a French translation of the book, which appeared in 1901.

Couturat had become Cantor's main philosophical supporter in France (where the point set topology was already well received); the book *De l'infini mathématique* (1896) recounted in great detail all aspects of set theory. Russell had reviewed it for *Mind* in the following year (*Papers* 2, 59–67) and used some of its features in his next foundational studies. Cantor's emphasis on order-types was especially attractive, as he could connect them to different kinds of relation, which he had already deployed in his own book on geometry as a means of handling order and recognised as an important philosophical category.

The next major influences came from two Cambridge colleagues. Firstly, Russell's former tutor, Alfred North Whitehead (1861–1947), was working in applied mathematics, including Grassmann's methods, which he deployed in *A Treatise on Universal Algebra, with Applications* (1898). The title was unfortunate, for no unifying algebra was presented; various different ones were treated, including George Boole's algebra of logic. Following Grassmann, Whitehead called collections 'manifolds' and handled them in traditional terms of part-whole theory, not with Cantorian distinctions. Secondly, Russell's slightly younger colleague G.E. Moore (1873–1958) revolted against the neo-Hegelian tradition in 1899 and put forward a strongly realist alternative, which Russell soon adopted.

Armed with these new tools, between 1898 and 1900 Russell tried out books on the foundations of mathematics (Griffin 1991, esp. Chapter 7). Using methods of reasoning and proof including Boole's and Grassmann's algebras, he explored 'the fundamental conceptions, and the necessary postulates of mathematics', including Whitehead's treatment of finite cardinals as extensional manifolds (*Papers* 3, 155–305). These efforts were soon followed by a much longer account of arithmetic, continuous quantities, and aspects of mechanics, with order and series given great prominence and relations in close attendance (*Papers* 3, 9–180). Whitehead remained significant; and Cantor was much more evident than before, not only for order but also on continuity and the transfinite numbers. However, Peano was not yet in sight.

Then Russell and Whitehead went to Paris late in July 1900 for the International Congress of Philosophy. The visit turned out to be a crucial experience for both men.

### III. FRIDAY 3 AUGUST 1900, AND RUSSELL'S CONCEPTION OF LOGICISM

The decisive event was a morning given over to the main Peanist quartet; Peano and Padoa were present in person, while organiser Couturat read contributions from Burali-Forti and Pieri. The magic moment came perhaps around 10.00. Peano had spoken about correct means of forming definitions in mathematical theories, and had emphasised the need for individuating the notion of 'the' when defining 'the class such that ...'. In the audience was the algebraic logician Ernst Schröder, who rejected the need for such fuss; in his post-Boolean theory classes were definable from nouns and adjectives alone, and treated part-whole style. However, Peano held his ground, and the young Russell must have realised that subtleties were involved, which he needed to learn.

In his autobiography, Russell tells us what happened next: he received all of Peano's publications at once in Paris 'and immediately read them all', and then wrote a book during the rest of the year (*Auto* 1, 145; also in *MPD*, 72–3). Luckily, he kept its manuscript, so that we can see that the story is absurdly wrong; the writing and re-writing lasted until 1902 (Grattan-Guinness 1997). Firstly, he did not receive most of the Peanist writings for a month, during which time he proof-read his book on Leibniz. When he read them he learned mathematical logic; but he also noted that the Peanists had not extended their logic to relations, so he produced the necessary theory and published it as a paper in 1901 in Peano's *Rivista* (*Papers* 3, 310–49, 618–27). He used the main techniques, especially set theory rather than part-whole collections, to revise his treatment of continuous quantities (where he defined irrational numbers as classes of rational numbers less than some given one and without upper or lower limit), various aspects of set theory, order and relations, the differential and integral calculus, and metrical, descriptive, and projective geometry (with a quite different flavour from the earlier book). Four large Parts of a new book were produced; however, the foundations, especially the place of logic with relations and definitions of integers, were not formed.

The foundations came early in 1901. With his axiom system for arithmetic, Peano had reduced the foundations of the Weierstrassian edifice to three indefinables for the integers; Russell now proposed to define them nominally in terms of classes of similar classes, imitating Cantor's process of double abstraction but without its idealist character. Thus, 0 was the class containing the empty class, 1 the class of classes similar to that containing 0, 2 the class of classes similar to that containing 0 and 1, and so on, up to and including Cantor's transfinite numbers. A valuable feature was his clear distinction of 0, the empty class and literally nothing (*POM*, 75), a tri-distinction which had plagued mathematicians and philosophers for centuries, even Dedekind and Cantor. Ordinals were defined analogously as classes of well-ordered classes.

This use of set theory led Russell to reject the Peanist strategy of dividing logical notions from mathematical ones. Since set theoretical ones could appear under either heading, there was *no* dividing line: mathematical logic (with relations) alone could *subsume* all mathematical notions, objects as well as methods of reasoning. This was his logicism, which he articulated in the opening two parts of the new book during 1901 and 1902. More precisely, as he put it in the opening section there (*POM*, 3),

1. Pure Mathematics is the class of all propositions of the form '*p* implies *q*', where *p* and *q* are propositions each containing at least one or more variables, the same in the two propositions, and neither *p* nor *q* contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

The implicational form was crucial to his position; it may have come to him from noting the importance of necessary and sufficient conditions for the truth of theorems as assumed by and after Cauchy, and especially from considering his material already written on the hypothetical character of geometries given the legitimacy of non-Euclidean versions (p. 430). In addition, his policy of not restricting the range over which variables could range required that antecedent

conditions be imposed on each occasion (' $p$ ' over propositions, ' $x$ ' over real numbers, or whatever). The alliance with 'pure' mathematics was a non-standard use of the adjective.

#### IV. THE PRINCIPLES OF MATHEMATICS NEWLY FORMED

The first part of the new book began with the definition of logicism quoted above and continued with a detailed though largely non-symbolic account of mathematical logic including relations, classes and the mysteriousness of nothing, and quantification. The latter led Russell to focus on notions denoted by little words, especially the sextet '*all, every, any, a, some and the*' (*POM*, 72–3). Words such as 'proposition', 'propositional function', 'variable', 'term', 'entity' and 'concept' denoted *extra*-linguistic notions, while pieces of language indicating them included 'letter', 'symbol', 'sentence' and 'proper name' (Vuillemin 1968). A word 'indicated' a concept which (might) 'denote' a term (Richards 1980).

A related problem was denoting phrases (see Hylton in this volume). Mathematics motivated the need, for logicism required the expression of mathematical functions such as  $x^2$ , which Russell called 'denoting functions', in terms of propositional functions. It had been decreed by Cauchy and accepted by his successors that in mathematical analysis and connected topics functions should be single-valued so as to allow, for example, unique specification of the derivative (if it existed); hence Russell was concerned primarily with 'definite descriptions' (to use his own later name) rather than indefinite ones. However, he found no satisfactory theory to present in his book.

Still more serious was another matter. While developing set theory, Russell applied Cantor's power-class construction to the class of "all" classes and deployed identity as the attempted isomorphism. Thus, he came to consider the class  $C$  of all classes which do not belong to themselves, and Cantor's proof of the greater cardinality of the power-class now came out as the logical disaster that  $C$  belonged to itself *if and only if* it did not do so. This was a *double* contradiction, not just the single contradiction as used in, for example, proof by reduction to the absurd. He described 'the contradiction' in the new book (*POM*, Chapter 10); later he added an appendix proposing a solution, but he soon saw that it did not work.

The seventh and last part of the book was put together last, largely out of the manuscript finished shortly before Paris. Treating some aspects of dynamics, Russell drew upon the continuity of space as established in the treatment of irrational numbers, with geometry providing the environment; then within it 'rational Dynamics' was laid out as 'a branch of pure mathematics' in his implicational sense of the adjective, 'which introduces its subject-matter by definition, not by observation of the actual world' (p. 467). He then tried to lay out causal chains as implications, but assumed that 'from a sufficient [finite] number of events at a sufficient number of moments, one or more events at one or more moments can be inferred' (p. 478). One would have thought that an assiduous student of the finite and infinite would not commit such an elementary blunder. In any case, the link to logicism seems rather tenuous, especially to so-called 'pure' mathematics; for example, why dynamics but no statics, or mathematical physics?

In May 1902, Russell sent off his manuscript to Cambridge University Press for printing, under the title *The Principles of Mathematics*. In those happy days of book production he *then* (re-)read much of the pertinent literature, changing the text in places and adding most of the many footnotes. His reading included the main books of Gottlob Frege (1848–1925), from which he found that he had been anticipated in both his logicistic thesis (though asserted by Frege only of arithmetic and some mathematical analysis) and certain features of mathematical logic. So in June he wrote to Frege and told him of the paradox, which seemed to affect both of their systems. In reply Frege agreed, and attempted a repair which, like Russell's, failed. In later letters (published in Frege 1976, 217–51), they also discussed various features of logic, denoting and other topics, and Russell also revised on proof a few passages in his text. For example, a weak discussion of the little words 'a' and 'one' in arts. 128 and 132 was replaced by a warning that the distinction between 'one involved in *one term* or *a class*' should not be confused with the cardinal number one.

Frege also sent to Russell several papers and booklets, and Russell wrote another appendix to his book in the autumn of 1902 reviewing Frege's achievements in some detail. However, as he stated very clearly and honestly in the preface to his book, 'If I should have become acquainted sooner with the work of Professor Frege, I should have owed a great deal to him, but as it is I arrived independently at many results which he had already established' (*POM*, xviii).

Table 1. Summary by Parts of Russell's *The principles of mathematics* (1903). The first column indicates the numbers of chapters and pages.

Part	Summary of main contents
1: 'The indefinables of mathematics'; 10, 105	'Definition of pure mathematics'; 'Symbolic logic', 'Implication and formal implication'; 'Proper names, adjectives and verbs', 'Denoting'; 'Classes', 'Propositional functions', 'The variable', 'Relations'; 'The contradiction'
2: 'Number'; 8, 43	Cardinals, definition and operations; 'Finite and infinite'; Peano axioms; Numbers as classes; 'Whole and part', 'Infinite wholes'; 'Ratios and fractions'
3: 'Quantity'; 5, 40	'The meaning of magnitude'; 'The range of quantity', numbers and measurement; 'Zero'; 'Infinite, the infinitesimal, and continuity'
4: 'Order'; 8, 58	Series, open and closed; 'Meaning of order', 'Asymmetrical relations', 'Difference of sense and of sign'; 'Progressions and ordinal numbers', 'Dedekind's theory of number'; 'Distance'
5: 'Infinity and continuity'; 12, 110	'Correlation of series'; real and irrational numbers, limits; continuity, Cantor's and ordinal; transfinite cardinals and ordinals; calculus; infinitesimals, infinite and the continuum
6: 'Space'; 9, 91	'Complex numbers'; geometries, projective, descriptive, metrical; Definitions of spaces; continuity, Kant; Philosophy of points
7: 'Matter and motion'; 7, 34	'Matter'; 'Motion', definition, absolute and relative, Newton's laws; 'Causality', 'Definition of dynamical world', 'Hertz's dynamics'
Appendix A: 23	Frege on logic and arithmetic
Appendix B: 6	'The doctrine of types'

Nevertheless, some commentators grossly exaggerate the extent of Frege's influence on Russell, both then and later.

The book finally appeared in May 1903. Table 1 summarises its main mathematical contents by part.

## V. COLLABORATION BUT INDECISION

Russell had publicised his new interest with a lecture course at Trinity College in the winter of 1901–2. One of his select audience

was Whitehead, who had begun to rework parts of Cantor's theory of infinitely large numbers in algebraic form. He published four papers in the *American Journal of Mathematics*, just over 100 pages in total length. He chose this venue because the editor of the journal was Frank Morley, a former fellow student at Trinity in the mid 1880s and by then professor at the Johns Hopkins University. Whitehead also found Russell's logicism a more clearly focused programme than his own investigations, and gradually their conversations turned into a formal collaboration to write a successor volume to *The Principles* (Lowe 1985). He gave courses himself at Cambridge on occasion; one is noted in §5. They were not often together, for he was living at Grantchester near Cambridge while in 1905 Russell built himself a house at Bagley Wood near Oxford.

Another student of his course, and undergraduate at the time, was Philip Jourdain (1879–1919). After graduation he worked on set theory and logic and launched an extensive correspondence with Russell (Grattan-Guinness 1977). In addition, the mathematician G.H. Hardy (1877–1947) was just starting his career with some papers in set theory and kept in quite close touch with Russell (Grattan-Guinness 1992); for example, he reviewed *The principles* perceptively, including pointing out the blunder in mechanics mentioned above (Hardy 1903). Finally, Moore was sympathetic to the enterprise; he moved to Edinburgh in 1904 for five years. Apart from Jourdain, all these associates were Apostles, like Russell himself.

As regards the technical work required, Russell gave much attention to the paradox, which he realised was very serious. He collected other paradoxes, or at least gave paradoxical status to certain results known previously (Garcia-diego 1992). Two important ones concerned the largest possible infinite cardinal and ordinal numbers ( $N$ , say); assumption of either of their existences led to contradictions such as  $N = N$  and  $N > N$ . Cantor had known both paradoxes but published neither; he told Jourdain that  $N$  belonged to the absolute infinite, *beyond* the actual infinite, and not a place for mankind to tread (Grattan-Guinness 1971, 115–16). Russell named the ordinal paradox after Burali-Forti, a name which has endured even though (Burali-Forti 1897) had not made such a claim but instead had exhibited an order-type for which trichotomy between ordinals did not apply. Curiously, around the same time, the American mathematician E.H. Moore (1862–1932), a close spectator of foundational studies,