Practical Extrapolation Methods

Theory and Applications

AVRAM SIDI
Technion–Israel Institute of Technology

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1

The Richardson Extrapolation Process

1.1 Introduction and Background

In many problems of practical interest, a given infinite sequence $\{A_n\}$ can be related to a function $A(y)$ that is known, and hence is computable, for $0 < y \leq b$ with some $b > 0$, the variable y being continuous or discrete. This relation takes the form $A_n = A(y_n),\ n = 0, 1, \ldots,$ for some monotonically decreasing sequence $\{y_n\} \subset (0, b]$ that satisfies $\lim_{n\to\infty} y_n = 0$. Thus, in case $\lim_{y\to 0+} A(y) = A,\ \lim_{n\to\infty} A_n = A$ as well. Consequently, computing $\lim_{n\to\infty} A_n$ amounts to computing $\lim_{y\to 0+} A(y)$ in such a case, and this is precisely what we want to do.

Again, in many cases of interest, the function $A(y)$ may have a well-defined expansion for $y \to 0+$ whose *form* is known. For example – and this is the case we treat in this chapter – $A(y)$ may satisfy for some positive integer s

$$A(y) = A + \sum_{k=1}^{s} \alpha_k y^{\sigma_k} + O(y^{\sigma_{s+1}}) \text{ as } y \to 0+, \tag{1.1.1}$$

where $\sigma_k \neq 0,\ k = 1, 2, \ldots, s+1$, and $\Re\sigma_1 < \Re\sigma_2 < \cdots < \Re\sigma_{s+1}$, and where α_k are constants *independent of* y. Obviously, $\Re\sigma_1 > 0$ guarantees that $\lim_{y\to 0+} A(y) = A$. When $\lim_{y\to 0+} A(y)$ does not exist, A is the *antilimit* of $A(y)$ for $y \to 0+$, and in this case $\Re\sigma_i \leq 0$ at least for $i = 1$. If (1.1.1) is valid for all $s = 1, 2, 3, \ldots,$ and $\Re\sigma_1 < \Re\sigma_2 < \cdots$, with $\lim_{k\to\infty} \Re\sigma_k = +\infty$, then $A(y)$ has the true asymptotic expansion

$$A(y) \sim A + \sum_{k=1}^{\infty} \alpha_k y^{\sigma_k} \text{ as } y \to 0+, \tag{1.1.2}$$

whether the infinite series $\sum_{k=1}^{\infty} \alpha_k y^{\sigma_k}$ converges or not. (In most cases of interest, this series diverges strongly.) The σ_k are assumed to be known, but the coefficients α_k need not be known; generally, the α_k are not of interest to us. We are interested in finding A whether it is the limit or the antilimit of $A(y)$ for $y \to 0+$.

Suppose now that $\Re\sigma_1 > 0$ so that $\lim_{y\to 0+} A(y) = A$. Then A can be approximated by $A(y)$ with sufficiently small values of y, the error in this approximation being $A(y) - A = O(y^{\sigma_1})$ as $y \to 0+$ by (1.1.1). If $\Re\sigma_1$ is sufficiently large, $A(y)$ can approximate A well even for values of y that are not too small. If this is not the case, however, then we may have to compute $A(y)$ for very small values of y to obtain reasonably good approximations to A. Unfortunately, this straightforward idea of reducing y to very small values is not always applicable. In most cases of interest, computing $A(y)$ for very small values of y either is very costly or suffers from loss of significance in finite-precision arithmetic. The deeper idea of the Richardson extrapolation, on the other hand, is to somehow eliminate the y^{σ_1} term from the expansion in (1.1.1) and to obtain a new approximation $A_1(y)$ to A whose error is $A_1(y) - A = O(y^{\sigma_2})$ as $y \to 0+$. Obviously, $A_1(y)$ will be a better approximation to A than $A(y)$ for small y since $\Re\sigma_2 > \Re\sigma_1$. In addition, if $\Re\sigma_2$ is sufficiently large, then we expect $A_1(y)$ to approximate A well also for values of y that are not too small, independently of the size of $\Re\sigma_1$. At this point, we mention only that the Richardson extrapolation is achieved by taking an appropriate "weighted average" of $A(y)$ and $A(\omega y)$ for some $\omega \in (0, 1)$. We give the precise details of this procedure in the next section.

From (1.1.1), it is clear that $A(y) - A = O(y^{\sigma_1})$ as $y \to 0+$, whether $\Re\sigma_1 > 0$ or not. Thus, the function $A_1(y)$ that results from the Richardson extrapolation can be a useful approximation to A for small values of y also when $\Re\sigma_1 \leq 0$, provided $\Re\sigma_2 > 0$. That is to say, $\lim_{y\to 0+} A_1(y) = A$ provided $\Re\sigma_2 > 0$ whether $\lim_{y\to 0+} A(y)$ exists or not. This is an additional fundamental and useful feature of the Richardson extrapolation.

In the following examples, we show how functions $A(y)$ exactly of the form we have described here come about naturally. In these examples, we treat the classic problems of computing π by the method of Archimedes, numerical differentiation by differences, numerical integration by the trapezoidal rule, summation of an infinite series that is used in defining the Riemann Zeta function, and the Hadamard finite parts of divergent integrals.

Example 1.1.1 The Method of Archimedes for Computing π The method of Archimedes for computing π consists of approximating the area of the unit disk (that is nothing but π) by the area of an inscribed or circumscribing regular polygon. If this polygon is inscribed in the unit disk and has n sides, then its area is simply $S_n = (n/2)\sin(2\pi/n)$. Obviously, S_n has the (convergent) series expansion

$$S_n = \pi + \frac{1}{2}\sum_{i=1}^{\infty} \frac{(-1)^i(2\pi)^{2i+1}}{(2i+1)!} n^{-2i}, \tag{1.1.3}$$

and the sequence $\{S_n\}$ is monotonically increasing and has π as its limit.

If the polygon circumscribes the unit disk and has n sides, then its area is $S_n = n\tan(\pi/n)$, and S_n has the (convergent) series expansion

$$S_n = \pi + \sum_{i=1}^{\infty} \frac{(-1)^i 4^{i+1}(4^{i+1}-1)\pi^{2i+1}B_{2i+2}}{(2i+2)!} n^{-2i}, \tag{1.1.4}$$

where B_k are the Bernoulli numbers (see Appendix D), and the sequence $\{S_n\}$ this time is monotonically decreasing and has π as its limit.

As the expansions given in (1.1.3) and (1.1.4) are also asymptotic as $n \to \infty$, S_n in both cases is analogous to the function $A(y)$. This analogy is as follows: $S_n \leftrightarrow A(y)$, $n^{-1} \leftrightarrow y$, $\sigma_k = 2k,\ k = 1, 2, \ldots$, and $\pi \leftrightarrow A$. The variable y is discrete and assumes the values $1/3, 1/4, \ldots$.

Finally, the subsequences $\{S_{2^m}\}$ and $\{S_{3\cdot 2^m}\}$ can be computed recursively without having to know π, their computation involving only square roots. (See Example 2.2.2 in Chapter 2.)

Example 1.1.2 Numerical Differentiation by Differences Let $f(x)$ be continuously differentiable at $x = x_0$, and assume that $f'(x_0)$, the first derivative of $f(x)$ at x_0, is needed. Assume further that the only thing available to us is $f(x)$, or a procedure that computes $f(x)$, for all values of x in a neighborhood of x_0.

If $f(x)$ is known in the neighborhood $[x_0 - a, x_0 + a]$ for some $a > 0$, then $f'(x_0)$ can be approximated by the centered difference $\delta_0(h)$ that is given by

$$\delta_0(h) = \frac{f(x_0 + h) - f(x_0 - h)}{2h}, \ 0 < h \le a. \tag{1.1.5}$$

Note that h here is a continuous variable. Obviously, $\lim_{h\to 0} \delta_0(h) = f'(x_0)$. The accuracy of $\delta_0(h)$ is quite low, however. When $f \in C^3[x_0 - a, x_0 + a]$, there exists $\xi(h) \in [x_0 - h, x_0 + h]$, for which the error in $\delta_0(h)$ satisfies

$$\delta_0(h) - f'(x_0) = \frac{f'''(\xi(h))}{3!} h^2 = O(h^2) \text{ as } h \to 0. \tag{1.1.6}$$

When the function $f(x)$ is continuously differentiable a number of times, the error $\delta_0(h) - f'(x_0)$ can be expanded in powers of h^2. For $f \in C^{2s+3}[x_0 - a, x_0 + a]$, there exists $\xi(h) \in [x_0 - h, x_0 + h]$, for which we have

$$\delta_0(h) = f'(x_0) + \sum_{k=1}^{s} \frac{f^{(2k+1)}(x_0)}{(2k+1)!} h^{2k} + R_s(h), \tag{1.1.7}$$

where

$$R_s(h) = \frac{f^{(2s+3)}(\xi(h))}{(2s+3)!} h^{2s+2} = O(h^{2s+2}) \text{ as } h \to 0. \tag{1.1.8}$$

The proof of (1.1.7) and (1.1.8) can be achieved by expanding $f(x_0 \pm h)$ in a Taylor series about x_0 with remainder.

The difference $\delta_0(h)$ is thus seen to be analogous to the function $A(y)$. This analogy is as follows: $\delta_0(h) \leftrightarrow A(y)$, $h \leftrightarrow y$, $\sigma_k = 2k, \ k = 1, 2, \ldots$, and $f'(x_0) \leftrightarrow A$.

When $f \in C^\infty[x_0 - a, x_0 + a]$, the expansion in (1.1.7) holds for all $s = 0, 1, \ldots$. As a result, we can replace it by the genuine asymptotic expansion

$$\delta_0(h) \sim f'(x_0) + \sum_{k=1}^{\infty} \frac{f^{(2k+1)}(x_0)}{(2k+1)!} h^{2k} \text{ as } h \to 0, \tag{1.1.9}$$

whether the infinite series on the right-hand side of (1.1.9) converges or not.

As is known, in finite-precision arithmetic, the computation of $\delta_0(h)$ for very small values of h is dominated by roundoff. The reason for this is that as $h \to 0$ both $f(x_0 + h)$ and $f(x_0 - h)$ tend to $f(x_0)$, which causes the difference $f(x_0 + h) - f(x_0 - h)$ to have fewer and fewer correct significant digits. Thus, it is meaningless to carry out the computation of $\delta_0(h)$ beyond a certain threshold value of h.

Example 1.1.3 Numerical Quadrature by Trapezoidal Rule Let $f(x)$ be defined on $[0, 1]$, and assume that $I[f] = \int_0^1 f(x)\,dx$ is to be computed by numerical quadrature. One of the simplest numerical quadrature formulas is the trapezoidal rule. Let $T(h)$ be the trapezoidal rule approximation to $I[f]$, with $h = 1/n$, n being a positive integer. Then, $T(h)$ is given by

$$T(h) = h\left[\frac{1}{2}f(0) + \sum_{j=1}^{n-1} f(jh) + \frac{1}{2}f(1)\right]. \tag{1.1.10}$$

Note that h for this problem is a discrete variable that takes on the values $1, 1/2, 1/3, \ldots$. It is well known that $T(h)$ tends to $I[f]$ as $h \to 0$ (or $n \to \infty$), whenever $f(x)$ is Riemann integrable on $[0, 1]$. When $f \in C^2[0, 1]$, there exists $\xi(h) \in [0, 1]$, for which the error in $T(h)$ satisfies

$$T(h) - I[f] = \frac{f''(\xi(h))}{12}h^2 = O(h^2) \text{ as } h \to 0. \tag{1.1.11}$$

When the integrand $f(x)$ is continuously differentiable a number of times, the error $T(h) - I[f]$ can be expanded in powers of h^2. For $f \in C^{2s+2}[0, 1]$, there exists $\xi(h) \in [0, 1]$, for which

$$T(h) = I[f] + \sum_{k=1}^{s} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]h^{2k} + R_s(h), \tag{1.1.12}$$

where

$$R_s(h) = \frac{B_{2s+2}}{(2s+2)!}f^{(2s+2)}(\xi(h))h^{2s+2} = O(h^{2s+2}) \text{ as } h \to 0. \tag{1.1.13}$$

Here B_p are the Bernoulli numbers as before. The expansion in (1.1.12) with (1.1.13) is known as the Euler–Maclaurin expansion (see Appendix D) and its proof can be found in many books on numerical analysis.

The approximation $T(h)$ is analogous to the function $A(y)$ in the following sense: $T(h) \leftrightarrow A(y)$, $h \leftrightarrow y$, $\sigma_k = 2k$, $k = 1, 2, \ldots$, and $I[f] \leftrightarrow A$.

Again, for $f \in C^{2s+2}[0, 1]$, an expansion that is identical in form to (1.1.12) with (1.1.13) exists for the midpoint rule approximation $M(h)$, where

$$M(h) = h\sum_{j=1}^{n} f(jh - \tfrac{1}{2}h). \tag{1.1.14}$$

This expansion is

$$M(h) = I[f] + \sum_{k=1}^{s} \frac{B_{2k}(\frac{1}{2})}{(2k)!}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]h^{2k} + R_s(h), \tag{1.1.15}$$

where, again for some $\xi(h) \in [0, 1]$,

$$R_s(h) = \frac{B_{2s+2}(\frac{1}{2})}{(2s+2)!}f^{(2s+2)}(\xi(h))h^{2s+2} = O(h^{2s+2}) \text{ as } h \to 0. \tag{1.1.16}$$

Here $B_p(x)$ is the Bernoulli polynomial of degree p and $B_{2k}(\frac{1}{2}) = -(1 - 2^{1-2k})B_{2k}$, $k = 1, 2, \ldots$.

When $f \in C^{\infty}[0, 1]$, both expansions in (1.1.12) and (1.1.15) hold for all $s = 0, 1, \ldots$. As a result, we can replace both by genuine asymptotic expansions of the form

$$Q(h) \sim I[f] + \sum_{k=1}^{\infty} c_k h^{2k} \quad \text{as } h \to 0, \tag{1.1.17}$$

where $Q(h)$ stands for $T(h)$ or $M(h)$, and c_k is the coefficient of h^{2k} in (1.1.12) or (1.1.15). Generally, when $f(x)$ is not analytic in $[0, 1]$, or even when it is analytic there but is not entire, the infinite series $\sum_{k=1}^{\infty} c_k h^{2k}$ in (1.1.17) diverges very strongly.

Finally, by $h = 1/n$, the computation of $Q(h)$ for very small values of h involves a large number of integrand evaluations and hence is very costly.

Example 1.1.4 Summation of the Riemann Zeta Function Series Let $A_n = \sum_{m=1}^{n} m^{-z},\ n = 1, 2, \ldots$. When $\Re z > 1$, $\lim_{n\to\infty} A_n = \zeta(z)$, where $\zeta(z)$ is the Riemann Zeta function. For $\Re z \le 1$, on the other hand, $\lim_{n\to\infty} A_n$ does not exist. Actually, the infinite series $\sum_{m=1}^{\infty} m^{-z}$ is taken as the definition of $\zeta(z)$ for $\Re z > 1$. With this definition, $\zeta(z)$ is an analytic function of z for $\Re z > 1$. Furthermore, it can be continued analytically to the whole z-plane with the exception of the point $z = 1$, where it has a simple pole with residue 1.

For all $z \neq 1$, i.e., whether $\lim_{n\to\infty} A_n$ exists or not, we have the well-known asymptotic expansion (see Appendix E)

$$A_n \sim \zeta(z) + \frac{1}{1-z} \sum_{i=0}^{\infty} (-1)^i \binom{1-z}{i} B_i n^{-z-i+1} \quad \text{as } n \to \infty, \tag{1.1.18}$$

where B_i are the Bernoulli numbers as before and $\binom{a}{i}$ are the binomial coefficients. We also recall that $B_3 = B_5 = B_7 = \cdots = 0$, and that the rest of the B_i are nonzero.

The partial sum A_n is thus analogous to the function $A(y)$ in the following sense: $A_n \leftrightarrow A(y)$, $n^{-1} \leftrightarrow y$, $\sigma_1 = z - 1$, $\sigma_2 = z$, $\sigma_k = z + 2k - 5,\ k = 3, 4, \ldots$, and $\zeta(z) \leftrightarrow A$ provided $z \neq -m + 1,\ m = 0, 1, 2, \ldots$. Thus, $\zeta(z)$ is the limit of $\{A_n\}$ when $\Re z > 1$, and its antilimit otherwise, provided $z \neq -m + 1,\ m = 0, 1, 2, \ldots$. Obviously, the variable y is now discrete and takes on the values $1, 1/2, 1/3, \ldots$.

Note also that the infinite series on the right-hand side of (1.1.18) is strongly divergent.

Example 1.1.5 Numerical Integration of Periodic Singular Functions Let us now consider the integral $I[f] = \int_0^1 f(x)\,dx$, where $f(x)$ is a 1-periodic function that is infinitely differentiable on $(-\infty, \infty)$ except at the points $t + k,\ k = 0, \pm1, \pm2, \ldots$, where it has logarithmic singularities, and can be written in the form $f(x) = g(x) \log|x - t| + \tilde{g}(x)$ when $x, t \in [0, 1]$. For example, with $u \in C^{\infty}(-\infty, \infty)$ and periodic with period 1, and with c some constant, $f(x) = u(x) \log(c|\sin \pi(x - t)|)$ is such a function. For this $f(x)$, we have $g(t) = u(t)$ and $\tilde{g}(t) = u(t) \log(\pi c)$. Sidi and Israeli [310] derived the "corrected" trapezoidal rule approximation

$$T(h; t) = h \sum_{i=1}^{n-1} f(t + ih) + \tilde{g}(t)h + g(t)h \log\left(\frac{h}{2\pi}\right), \quad h = 1/n, \tag{1.1.19}$$

for $I[f]$, and showed that $T(h;t)$ has the asymptotic expansion

$$T(h;t) \sim I[f] - 2\sum_{k=1}^{\infty} \frac{\zeta'(-2k)}{(2k)!} g^{(2k)}(t) h^{2k+1} \quad \text{as } h \to 0. \tag{1.1.20}$$

Here $\zeta'(z) = \frac{d}{dz}\zeta(z)$. (See Appendix D.)

The approximation $T(h;t)$ is analogous to the function $A(y)$ in the following sense: $T(h;t) \leftrightarrow A(y)$, $h \leftrightarrow y$, $\sigma_k = 2k+1$, $k = 1, 2, \ldots$, and $I[f] \leftrightarrow A$. In addition, y takes on the discrete values $1, 1/2, 1/3, \ldots$.

Example 1.1.6 Hadamard Finite Parts of Divergent Integrals Consider the integral $\int_0^1 x^\rho g(x)\,dx$, where $g \in C^\infty[0,1]$ and ρ is generally complex such that $\rho \neq -1, -2, \ldots$. When $\Re\rho > -1$, the integral exists in the ordinary sense. In case $g(0) \neq 0$ and $\Re\rho \leq -1$, the integral does not exist in the ordinary sense since $x^\rho g(x)$ is not integrable at $x = 0$, but its Hadamard finite part exists, as we mentioned in Example 0.2.4. Let us define $Q(h) = \int_h^1 x^\rho g(x)\,dx$. Obviously, $Q(h)$ is well-defined and computable for $h \in (0,1]$. Let m be any nonnegative integer. Then, there holds

$$Q(h) = \int_h^1 x^\rho \left[g(x) - \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} x^i \right] dx + \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} \frac{1 - h^{\rho+i+1}}{\rho+i+1}. \tag{1.1.21}$$

Now let $m > -\Re\rho - 1$. Expressing the integral term in (1.1.21) in the form $\int_h^1 = \int_0^1 - \int_0^h$, using the fact that

$$g(x) - \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} x^i = \frac{g^{(m)}(\xi(x))}{m!} x^m, \quad \text{for some } \xi(x) \in (0, x),$$

and defining

$$I(\rho) = \int_0^1 x^\rho \left[g(x) - \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} x^i \right] dx + \sum_{i=0}^{m-1} \frac{1}{\rho+i+1} \frac{g^{(i)}(0)}{i!}, \tag{1.1.22}$$

and $\|g^{(m)}\| = \max_{0 \leq x \leq 1} |g^{(m)}(x)|$, we obtain from (1.1.21)

$$Q(h) = I(\rho) - \sum_{i=0}^{m-1} \frac{g^{(i)}(0)}{i!} \frac{h^{\rho+i+1}}{\rho+i+1} + R_m(h); \quad |R_m(h)| \leq \frac{\|g^{(m)}\|}{m!} \frac{h^{\Re\rho+m+1}}{\Re\rho+m+1}, \tag{1.1.23}$$

[Note that, with $m > -\Re\rho - 1$, the integral term in (1.1.22) exists in the ordinary sense and $I(\rho)$ is independent of m.] Since m is also arbitrary in (1.1.23), we conclude that $Q(h)$ has the asymptotic expansion

$$Q(h) \sim I(\rho) - \sum_{i=0}^{\infty} \frac{g^{(i)}(0)}{i!} \frac{h^{\rho+i+1}}{\rho+i+1} \quad \text{as } h \to 0. \tag{1.1.24}$$

Thus, $Q(h)$ is analogous to the function $A(y)$ in the following sense: $Q(h) \leftrightarrow A(y)$, $h \leftrightarrow y$, $\sigma_k = \rho + k$, $k = 1, 2, \ldots$, and $I(\rho) \leftrightarrow A$. Of course, y is a continuous variable in this case. When the integral exists in the ordinary sense, $I(\rho) = \lim_{h\to 0} Q(h)$; otherwise, $I(\rho)$ is the Hadamard finite part of $\int_0^1 x^\rho g(x)\,dx$ and serves as the antilimit of $Q(h)$

as $h \to 0$. Finally, $I(\rho) = \int_0^1 x^\rho g(x)\,dx$ is analytic in ρ for $\Re\rho > -1$ and, by (1.1.22), can be continued analytically to a meromorphic function with simple poles possibly at $\rho = -1, -2, \ldots$. Thus, the Hadamard finite part is nothing but the analytic continuation of the function $I(\rho)$ that is defined via the convergent integral $\int_0^1 x^\rho g(x)\,dx$, $\Re\rho > -1$, to values of ρ for which $\Re\rho \le -1$, $\rho \ne -1, -2, \ldots$.

Before going on, we mention that many of the developments of this chapter are due to Bulirsch and Stoer [43], [45], [46]. The treatment in these papers assumes that the σ_k are real and positive. The case of generally complex σ_k was considered recently in Sidi [298], where the function $A(y)$ is allowed to have a more general asymptotic behavior than in (1.1.2). See also Sidi [301].

1.2 The Idea of Richardson Extrapolation

We now go back to the function $A(y)$ discussed in the second paragraph of the preceding section. We do not assume that $\lim_{y\to 0+} A(y)$ necessarily exists. We recall that, when it exists, this limit is equal to A in (1.1.1) ; otherwise, A there is the antilimit of $A(y)$ as $y \to 0+$. Also, the nonexistence of $\lim_{y\to 0+} A(y)$ immediately implies that $\Re\sigma_i \le 0$ at least for $i = 1$.

As mentioned in the third paragraph of the preceding section, $A(y) - A = O(y^{\sigma_1})$ as $y \to 0+$, and we would like to eliminate the y^{σ_1} term from (1.1.1) and thus obtain a new approximation to A that is better than $A(y)$ for $y \to 0+$. Let us pick a constant $\omega \in (0, 1)$, and set $y' = \omega y$. Then, from (1.1.1) we have

$$A(y') = A + \sum_{k=1}^{s} \alpha_k \omega^{\sigma_k} y^{\sigma_k} + O(y^{\sigma_{s+1}}) \text{ as } y \to 0+ . \tag{1.2.1}$$

Multiplying (1.1.1) by ω^{σ_1} and subtracting from (1.2.1), we obtain

$$A(y') - \omega^{\sigma_1} A(y) = (1 - \omega^{\sigma_1})A + \sum_{k=2}^{s} (\omega^{\sigma_k} - \omega^{\sigma_1})\alpha_k y^{\sigma_k} + O(y^{\sigma_{s+1}}) \text{ as } y \to 0+ . \tag{1.2.2}$$

Obviously, the term y^{σ_1} is missing from the summation in (1.2.2). Dividing both sides of (1.2.2) by $(1 - \omega^{\sigma_1})$, and identifying

$$A(y, y') = \frac{A(y') - \omega^{\sigma_1} A(y)}{1 - \omega^{\sigma_1}} \tag{1.2.3}$$

as the new approximation to A, we have

$$A(y, y') = A + \sum_{k=2}^{s} \frac{\omega^{\sigma_k} - \omega^{\sigma_1}}{1 - \omega^{\sigma_1}} \alpha_k y^{\sigma_k} + O(y^{\sigma_{s+1}}) \text{ as } y \to 0+, \tag{1.2.4}$$

so that $A(y, y') - A = O(y^{\sigma_2})$ as $y \to 0+$, as was required. It is important to note that (1.2.4) is exactly of the form (1.1.1) with $A(y)$ and the α_k replaced by $A(y, y')$ and the $\frac{\omega^{\sigma_k} - \omega^{\sigma_1}}{1 - \omega^{\sigma_1}} \alpha_k$, respectively.