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Abstract Regular Polytopes

PETER McMULLEN
University College London

EGON SCHULTE
Northeastern University

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Classical Regular Polytopes

Our purpose in this introductory chapter is to set the scene for the rest of the book. We shall do this by briefly tracing the historical development of the subject. There are two main reasons for this. First, we wish to recall the historical traditions which lie behind current research. It is all too easy to lose track of the past, and it is as true in mathematics as in anything else that those who forget history may be compelled to repeat it. But perhaps more important is the need to base what we do firmly in the historical tradition. A tendency in mathematics to greater and greater abstractness should not lead us to abandon our roots. In studying abstract regular polytopes, we shall always bear in mind the geometric origins of the subject. We hope that this introductory survey will help the reader to find a firm basis from which to view the modern subject.

The chapter has four sections. In the first, we provide an historical sketch, leading up to the point at which the formal material of Chapter 2 begins. The second is devoted to an outline of the theory of regular convex polytopes, which provide so much of the motivation for the abstract definitions which we subsequently adopt. In the third, we treat various generalizations of regular polytopes, mainly in ordinary euclidean space, including the classical regular star-polytopes. In the fourth, we introduce regular maps, which are the first examples of abstract regular polytopes, although the examples considered here occur before the general theory was formulated.

1A The Historical Background

Regular polytopes emerge only gradually out of the mists of history. Apart from certain planar figures, such as squares and triangles, the cube, in the form of a die, was probably the earliest known to man. Gamblers would have used dice from the earliest days, and a labelled example helped linguists to work out the Etruscan words for "one" up to "six". The Etruscans also had dodecahedral dice; examples date from before 500BCE, and may even be much earlier. The other three "platonic" solids appear not to have been employed in gambling; two out of the three do not roll well in any case.

However, it is only somewhat later that the regular solids were studied for their own sakes, and the leap from them to the regular star-polyhedra, analogous to that from pentagon to pentagram, had to await the later middle ages. The nineteenth century first saw regular polytopes of higher dimension, but the real flowering of what is, in origin, one of the oldest branches of mathematics occurred only in the twentieth century.

In this section, we shall give a brief outline of the historical background to the theory of regular polytopes. This is not intended to be totally comprehensive, although we have attempted to give the salient features of more than two millenia of investigations in the subject.

The Classical Period

Before the Greeks

As we have already said, the cube was probably the first known regular polyhedron; certainly it was well known before the ideas of geometry and symmetry had themselves been formulated. Curiously, though, stone balls incised in ways that illustrate all the symmetry groups of the regular polyhedra were discovered in Scotland in the nineteenth century; they appear to date from the first half of the third millenium BCE (see [137, Chapter 7]).

The Egyptians were also aware of the regular tetrahedron and octahedron. As an aside, we pose the following question. Many attempts have been made to explain why the pyramids are the shape they are, or, more specifically, why the ratio of height to base of a square pyramid is always roughly the same particular number. In particular, such explanations often manage to involve π in some practical way, such as measurements by rolling barrels – the Egyptians' theoretical value $\frac{256}{81} \approx 3.16$ for π was fairly accurate. Is it possible, though, that a pyramid was intended to be half an octahedron? The actual angles of slopes of the four proper pyramids at Giza vary between $50^\circ 47'$ and $54^\circ 14'$; the last is only a little short of half the dihedral angle of the octahedron, namely, $\arccos(1/\sqrt{3}) \approx 54^\circ 44'$.

The Early Greeks

Despite various recent claims to the contrary, it seems clear that the Greeks were the first to conceive of mathematics as we now understand it. (The mere listing of, for example, Pythagorean triples does not constitute mathematics; a proof of Pythagoras's theorem obviously does.) According to Proclus (412–485CE), the discovery of the five regular solids was attributed by Eudemus to Pythagoras (c582–c500BCE) himself; the fact that a point of the plane is exactly surrounded by six equilateral triangles, four squares or three regular hexagons (these giving rise to the three regular tilings of the plane) was also known to his followers. They knew as well of the regular pentagram, apparently regarding it as a symbol of health; it has been suggested that this also gave them their first example of incommensurability. (For a fuller account of the origin of the regular solids, consult [438].)

The regular solids in ordinary space were named after Plato (Aristocles son of Ariston, 427–347BCE) by Heron; this seems to be one of the earliest mathematical misattributions. Indeed, their first rigorous mathematical treatment was by Theaetetus (c415–369BCE, when he was killed in battle). In his *Timaeus*, Plato does discuss the regular solids, but while his enthusiasm for and appreciation of the figures are obvious, it is also evident that his discussion falls short of a full mathematical investigation. However, one very perceptive idea does appear there. An equilateral triangle is regarded by him as formed from the six right-angled subtriangles into which it is split by its altitudes. The three solids with triangular faces (tetrahedron, octahedron, and icosahedron) are then built up from these subdivided triangles. This anticipates the construction of the Coxeter kaleidoscope of their reflexion planes by more than two millennia. But the general principle was not fully recognized by Plato; this is exhibited by his splitting of the square faces of the cube into four isosceles (instead of eight) triangles. Moreover, the dodecahedron is not seen in this way at all. In the *Timaeus*, one has the impression that the existence of the dodecahedron (identified with the universe) almost embarrasses Plato. The other four regular solids are identified with the four basic elements – tetrahedron = fire, octahedron = air, cube = earth, and icosahedron = water – in a preassumed scheme which is not at all scientific. (In the *Phaido*, amusingly, Plato also describes dodecahedra; they appear as stuffed leather balls made out of twelve multicoloured pentagonal pieces, an interesting near anticipation of some modern association footballs.)

To Plato's pupil Aristotle (384–322BCE) is attributed the mistaken assertion that the regular tetrahedron tiles ordinary space. Unfortunately, such was the high regard in which Aristotle was held in later times that his opinion was not challenged until comparatively recently, although its falsity could have been established at the time it was made.

Euclid

Euclid's *Elements* ($\Sigma\tau o\iota\chi\epsilon\iota\alpha$) is undoubtedly the earliest surviving true mathematics book, in the sense that it fully recognizes the characteristic mathematical paradigm of axiom–definition–theorem–proof. Until early in the twentieth century, parts of it, notably the first six Books ($\Sigma\chi o\lambda\iota\alpha$), provided, essentially unchanged, a fine introduction to basic geometry. It is unclear to what extent Euclid discovered his material or merely compiled it; our ignorance of Euclid himself extends to our being uncertain of more than that, as we are told by Proclus, he lived and worked in Alexandria at the time of Ptolemy I Soter (reigned 323–283BCE).

Of course, it is to Euclid that we look for the first rigorous account of the five regular solids; Proclus even claimed that *Elements* is designed to lead up to the discussion of them. Whether or not that contention can be justified, Book XIII is devoted to the regular solids. (Incidentally, this book and Book X are less than thoroughly integrated into the rest of the text, suggesting that they were incorporated from an already existing work, which may well have been written by Theaetetus himself.) The scholium (theorem) of that book demonstrates that there are indeed only five regular solids. The proof is straightforward, and (in essence) remains that still used: the angle at a vertex of a regular p-gon is $(1 - \frac{2}{p})\pi$, and so for q of them to fit around a vertex of a regular solid, one requires that $q(1 - \frac{2}{p})\pi < 2\pi$, or, in other words,

1A1 $$\frac{1}{p} + \frac{1}{q} > \frac{1}{2}.$$

Euclid did not phrase the result in quite this way, but this is what the proof amounts to.

Further results about the regular solids occur in Books XIV and XV (which were written around 300CE, and so were not by Euclid), such as their metrical properties, and in particular some anticipation of duality. The details, and modern explanations of the results, are described in [120].

Archimedes

We should also briefly mention here a contribution of Archimedes (c287–212BCE). The Archimedean polyhedra themselves are beyond the scope of this book. However, Archimedes did use regular polygons – actually, the 96-gon – to find his famous bounds $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The remainder of the many mathematical results of Archimedes are not relevant to the topic of this book, but their significance cannot be allowed to pass altogether unnoticed.

The Romans

The Romans were fine architects and engineers, but contributed less to mathematics. However, among the various works attributed to the great pagan philosopher Anicius Manlius Severinus Boetius (c480–524CE) (his name is usually miswritten as "Boethius") is a translation of most of the first three books of Euclid. Since he certainly translated Plato and Aristotle (with a view to reconciling them), this is a possibility which cannot lightly be dismissed.

The Mediaeval Period

The Early Middle Ages

The Christian Roman Empire got off to a poor start in its treatment of learning; under a decree of Emperor Theodosius I ("the Great") concerning pagan monuments, in around 389–391 Bishop Theophilus ordered that the great library of Alexandria (or, at least, that part in the Serapeum) be pillaged. (It is uncertain how much of the original library had survived to this time; it is said – though the event is disputed – that the larger part, the Brucheum, was burnt around 47BCE when Caesar set fire to the Egyptian fleet during the Roman civil wars. The later story of the Muslim destruction under 'Amr is of much more dubious provenance.) A little later, the last of the Alexandrine philosophers, the talented and beautiful Hypatia (c375–415), was flayed with oyster shells by a Christian mob at the instigation of Bishop Cyril (who was later canonized).

The attitude of the Byzantine (Eastern Roman) Empire to mathematics (and the other sciences) was distinctly ambiguous, alternating between encouragement and suppression. Justinian I (reigned 527–565) initially seemed supportive, but soon closed the

Academy at Athens in 529, although there was probably little resulting loss to mathematics. It is due to a few people in the ninth (particularly Leo “the mathematician”) and tenth centuries that we have the Greek manuscripts of Euclid which survive; the earliest dates from 888. Similarly, a tenth-century manuscript of Archimedes was reused in the twelfth century (a palimpsest) for religious texts; fortunately, the original mathematics can be recovered by modern techniques. It must be concluded that the Byzantines preserved rather than added to the corpus of knowledge.

The mathematical torch was also carried on by the Arabs, but they too seem to have added little to geometry, although their translation of Euclid helped to preserve it. (The contributions of the Islamic world to algebra are a quite different matter.) They had a good empirical knowledge of symmetry; in the Alhambra there are patterns which exemplify many of the seventeen possible planar symmetry groups (and the rest can be produced by slight modifications of some of the others).

The Later Middle Ages

From about the twelfth century, mathematical knowledge began seeping back into western Europe. Around the 1120s, Aethelard (Adelard) of Bath, known as “Philosophus Anglorum”, produced a translation of Euclid; while he knew Greek, this is more likely to have been from Arabic. (It was first printed in Venice in 1482 under the name of Campanus of Novara, with an unhelpful commentary, but the attribution to Aethelard is universally accepted.)

Rather later, Thomas Bradwardine “the Profound Doctor” (c1290–1349), Archbishop of Canterbury for just forty days after his consecration (he died of plague), systematically investigated star-polygons, obtaining $\{\frac{n}{d}\}$ by stellating $\{\frac{n}{d-1}\}$. (The notation will be explained later in the chapter.)

Although Kepler and Poinsot (see the following subsection) are credited with discovering the regular star-polyhedra in three dimensions, the polyhedron $\{\frac{5}{2}, 5\}$ was depicted in 1420 by Paolo Uccello (1397–1475), while $\{5, \frac{5}{2}\}$ occurs in an engraving of 1568 by Wenzel Jamnitzer (1508–85); however, it is unlikely that they fully appreciated the differences between these figures and others that they drew.

The Modern Period

Before Schläfli

Johannes Kepler (1571–1630) began the modern investigation of regular polytopes by his discovery of the two star-polyhedra $\{\frac{5}{2}, 5\}$ (strictly, perhaps, a rediscovery) and $\{\frac{5}{2}, 3\}$ (see [248, p. 122]). He also investigated various regular star-polygons, particularly the heptagons; for the latter, he showed that the side lengths of the three heptagons $\{7\}$, $\{\frac{7}{2}\}$ and $\{\frac{7}{3}\}$ inscribed in the unit circle are the roots of the equation

1A2 $$\lambda^6 - 7\lambda^4 + 14\lambda^2 - 7 = 0.$$

In a sense, Kepler stands on a cusp. The lingering effect of mediaeval (or perhaps even classical) thought on him shows in his attempt to relate the relative sizes of the orbits of the planets to the ratios of in- and circumradii of the regular polyhedra; later in his life he demonstrated that these orbits were ellipses.

The Greeks had proved that certain regular polygons, notably the pentagon, were constructible using ruler and compass alone. In 1796, the young Carl Friedrich Gauss (1777–1855) showed that, if a regular n-gon $\{n\}$ can be so constructed, then n is a power of 2 times a product of distinct Fermat primes, of the form

1A3 $$p = 2^{2^k} + 1$$

for some k; in fact, the condition is sufficient as well as necessary. The only known Fermat primes are those for $k = 0, 1, 2, 3, 4$; if there are no others, then an odd such n is a divisor of $2^{32} - 1$. In 1809, Louis Poinsot (1777–1859) rediscovered the first two regular star-polyhedra, and found their duals $\{5, \frac{5}{2}\}$ (again, perhaps really a rediscovery) and $\{3, \frac{5}{2}\}$ (see [342]); very soon afterwards, in 1811, Augustin Louis Cauchy (1789–1857) proved that the list of such regular star-polyhedra was now complete (see [76]).

Schläfli

At a time when very few mathematicians had any concept of working in higher dimensional spaces, Ludwig Schläfli (1814–95) discovered regular polytopes and honeycombs in four and more dimensions around 1850 (see [355, §17, 18]). In fact, he found all the groups of the regular polytopes whose symmetry groups are generated by reflexions in hyperplanes in euclidean spaces. But against all his evidence he refused to recognize the dual pair $\{\frac{5}{2}, 5\}$ and $\{5, \frac{5}{2}\}$ as “genuine” polyhedra (because they have non-zero genus), and so would not accept either the regular 4-polytopes which have these as facets or vertex-figures, even though calculating the spherical volumes of corresponding tetrahedra on the 3-sphere was a central part of his treatment.

From 1880 onwards, the regular polytopes in higher dimensions were rediscovered many times, beginning with Stringham [405]. We refer to [120, p. 144] for the relevant details. Edmund Hess [215] found the remaining regular star-polytopes, and S. L. van Oss [337] proved that the enumeration was complete. (For an argument avoiding consideration of each separate case, see [280] and Section 7D in this work.)

Coxeter

The subject of regular polytopes had gone into somewhat of a decline when it was taken up by H. S. M. (Donald) Coxeter (born 1907). His investigations and consolidation of the theory culminated in his famous book *Regular Polytopes* [120], whose first edition was published in 1948. His contributions are too numerous to list here individually, but we should at least mention Coxeter diagrams and the complete classification of the discrete euclidean reflexion groups among all Coxeter groups. We shall mention this latter material in Section 1B.

But Coxeter also pointed towards later developments of the theory. In particular, when J. F. Petrie (1907–72) (the inventor of the skew polygon which bears his name) found the two regular skew apeirohedra $\{4, 6\,|\,4\}$ and $\{6, 4\,|\,4\}$, he immediately found the third $\{6, 6\,|\,3\}$, and set the whole theory in a general context (see [105]). He also looked at regular maps and their automorphism groups, regarding the star-polyhedra as particular examples; he first observed that the Petrie polygons of a regular map themselves (usually) form another regular map (see [131, p. 112]). We shall provide an introduction to this area in Section 1D.

In 1975, Grünbaum (see [198]) gave the theory a further impetus. He generalized the regular skew polyhedra, by allowing skew polygons as faces as well as vertex-figures. He found eight more individual examples and twelve infinite families (with non-congruent realizations of isomorphic apeirohedra), and Dress [148, 150] completed the classification by finding the final case and establishing the completeness of the list. Again, we shall consider this work later, in the appropriate place (see Section 7E).

Finally, regular polytopes also formed the cradle of Tits's work on buildings (see [415–417]). Buildings of spherical type are the natural geometric counterparts of simple Lie groups of Chevally type. Regular polytopes, or, more generally, Coxeter complexes (see Sections 2C and 3A), occur here as fundamental structural components, namely, as the "apartments" of buildings. In a further generalization, Buekenhout [55, 58] introduced the notion of a diagram geometry to find a geometric interpretation for the twenty-six sporadic groups (see [14, 143, 244]). Although we shall not discuss buildings and diagram geometries in detail, the present book has nevertheless been considerably influenced by these developments.

History teaches us that the subject of regular polyhedra has shown an enormous potential for revival. One natural explanation is that the beauty of the geometric figures appeals to the artistic senses [20, 384].

1B Regular Convex Polytopes

We begin this section with a short discussion of convexity, which we shall need again in Chapter 5. For fuller details, we refer the reader to any one of a number of standard texts, for example [197, 357].

A subset K of n-dimensional euclidean space $\mathbb{E}^n$ is *convex* if, with each two of its points x and y, it contains the *line segment*

$$[xy] := \{(1-\lambda)x + \lambda y \mid 0 \leqslant \lambda \leqslant 1\}.$$

The intersection of convex sets is again convex, and so the *convex hull* $\operatorname{conv} S$ of a set $S \subseteq \mathbb{E}^n$ is well defined as the smallest convex set which contains S. The convex hull of a finite set of points is a *convex polytope*; in this section, we shall frequently drop the qualifying term "convex" and talk simply about a polytope. A polytope P is *k-dimensional*, or a *k-polytope*, if its affine hull is k-dimensional. Here, an *affine subspace* of $\mathbb{E}^n$ is a subset A which contains each line

$$xy := \{(1-\lambda)x + \lambda y \mid \lambda \in \mathbb{R}\}$$

between two points $x, y \in A$; the *affine hull* aff S of a subset S is similarly the smallest affine subspace of $\mathbb{E}^n$ which contains S.

Bear in mind that a non-empty affine subspace A is a translate of a unique linear subspace

$$L := A - A = A - x$$

for any $x \in A$; by definition $\dim A := \dim L$. The empty set $\emptyset$ is the affine subspace of dimension -1; it is also a polytope. We further refer to 2-polytopes as *polygons* and to 3-polytopes as *polyhedra*.

The simplest example of an n-polytope is an *n-simplex*, which is the convex hull of an affinely independent set of $n+1$ points. Here, a set $\{a_0, a_1, \ldots, a_n\}$ is *affinely independent* if, whenever $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are such that

$$\sum_{i=0}^{n} \lambda_i a_i = o, \quad \sum_{i=0}^{n} \lambda_i = 0,$$

then $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$; this is the natural extension of the notion of linear independence. (We use "o" to denote the zero vector.)

A hyperplane

$$H(u, \alpha) := \{x \in \mathbb{E}^n \mid \langle x, u \rangle = \alpha\}$$

supports a convex set K, with *outer normal* u, if

$$\alpha = \sup\{\langle x, u \rangle \mid x \in K\}.$$

The intersection $H(u, \alpha) \cap K$ is then an (*exposed*) *face* of K. An n-polytope P has faces of each dimension $0, \ldots, n-1$, which are themselves polytopes. Often, $\emptyset$ and P itself are counted as faces of P, called the *improper* faces; the other faces are *proper*. We write $\mathcal{P}(P) = \mathcal{P}$ for the family of all faces of P. The faces of dimensions 0, 1, $n-2$ and $n-1$ are also referred to as its *vertices, edges, ridges* and *facets*, respectively; more generally, a face of dimension j is called a *j-face*.

The notation vert P is usual for the set of vertices of a polytope P; then $P = \operatorname{conv}(\operatorname{vert} P)$. If $v \in \operatorname{vert} P$, and if H is a hyperplane which strictly separates v from $\operatorname{vert} P \setminus \{v\}$, then $H \cap P$ is called a *vertex-figure* of P at v. In the cases we shall consider in the following, we may usually choose the vertices of the vertex-figure at v in some special way; traditionally, they are the midpoints of the edges through v, although we shall frequently violate the strict terms of the definition, and take the other vertices of the edges through v instead.

Before we proceed further, we list various properties of a convex n-polytope P, which will motivate many of the definitions we adopt in Chapter 2.

- $\mathcal{P}$ is a lattice, under the partial ordering $F \leqslant G$ if and only if $F \subseteq G$. The *meet* of two faces F and G is then $F \wedge G := F \cap G$, and the *join* $F \vee G$ is the (unique) smallest face of P which contains F and G.
- If $F < G$ are two faces of P with $\dim G - \dim F = 2$, then there are exactly two faces J of P such that $F < J < G$.

- For every two faces F, G of P with $F \leqslant G$, the *section*

$$G/F := \{J \in \mathcal{P} \mid F \leqslant J \leqslant G\}$$

  of $\mathcal{P}$ is isomorphic to the face-lattice of a polytope of dimension $\dim G - \dim F - 1$. (For $F = \emptyset$, we have $G/F = G$ by a minor abuse of notation; when $\dim F \geqslant 0$, proceed by induction, namely, by successive construction of vertex-figures.)

Two faces F and G of P are called *incident* if $F \leqslant G$ or $G \leqslant F$.

- If $\dim P \geqslant 2$, then $\mathcal{P}$ is *connected*, in the sense that any two proper faces F and G of P can be joined by a chain $F =: F_0, F_1, \dots, F_k := G$ of proper faces of P, such that F_{i-1} and F_i are incident for $i = 1, \dots, k$. Hence, $\mathcal{P}$ is *strongly connected*, in that the same is true for every section G/F of $\mathcal{P}$ such that $\dim G \geqslant \dim F + 3$.
- The boundary $\operatorname{bd} P$ of P is homeomorphic to an $(n-1)$-sphere; in particular, if $n \geqslant 3$, then $\operatorname{bd} P$ is simply connected.

We call two polytopes P and Q (*combinatorially*) *isomorphic* if their face-lattices $\mathcal{P}(P)$ and $\mathcal{P}(Q)$ are isomorphic, so that there is a one-to-one inclusion preserving correspondence between the faces of P and those of Q. Similarly, P and Q are *dual* if $\mathcal{P}(P)$ and $\mathcal{P}(Q)$ are anti-isomorphic, giving a one-to-one inclusion reversing correspondence between the faces of P and those of Q. The notation P^* for a dual of P will occur quite often.

A *flag* of an n-polytope P is a maximal subset of pairwise incident faces of P; thus, it is of the form $\{F_{-1}, F_0, \dots, F_{n-1}, F_n\}$, with

$$F_{-1} \subset F_0 \subset \cdots \subset F_{n-1} \subset F_n.$$

Here we introduce the conventions $F_{-1} := \emptyset$ and $F_n := P$ for an n-polytope P; the inclusions are strict, so that $\dim F_j = j$ for each $j = 0, \dots, n-1$. The improper faces $\emptyset$ and P are often omitted from the specification of a flag, since they belong to all of them. The family of flags of P is denoted $\mathcal{F}(P)$. Flags thus have the following properties.

- For each $j = 0, \dots, n-1$, there is a unique flag $\Phi^j \in \mathcal{F}(P)$ which differs from a given flag Φ in its j-face alone. Two such flags Φ and Φ^j are called *adjacent*, or, more exactly, *j-adjacent*.
- P is *strongly flag-connected*, in that for each two flags Φ and Ψ of P, there exists a chain $\Phi =: \Phi_0, \Phi_1, \dots, \Phi_k := \Psi$, such that Φ_{i-1} and Φ_i are adjacent for each $i = 1, \dots, k$, and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i = 1, \dots, k-1$.

The *symmetry group* $G(P)$ of P consists of the isometries g of $\mathbb{E}^n$ such that $Pg = P$.[†] Then P is called *regular* if $G(P)$ is transitive on the family $\mathcal{F}(P)$ of flags of P; this form of the definition seems to have been given first by Du Val in [156, p. 63].

Alternative definitions of regularity of an n-polytope are common in the literature. We list some of them here; a comprehensive discussion of this topic occurs in [279].

[†] In such algebraic contexts, we write maps after their arguments throughout the book. Compositions of maps thus occur in their natural order; that is, they are read from left to right. Note that these conventions are a change from those in some of our earlier publications.

- A polygon is regular if its edges have the same length, and the angles at its vertices are equal (or, its vertices lie on a circle).
- For $n \geqslant 3$, an n-polytope is regular if its facets are regular and congruent (or isomorphic), and its vertex-figures are isomorphic. (This formulation depends on Cauchy's rigidity theorem; see [242, p. 335].)
- For every n, an n-polytope P is regular if, for each $j = 0, \ldots, n-1$, its symmetry group $G(P)$ is transitive on the j-faces of P.

A *reflexion* R in $\mathbb{E}^n$ is an involutory isometry; it has a *mirror*

$$\{x \in \mathbb{E}^n \mid xR = x\}$$

of fixed points with which it is identified, so that the same notation R is employed for it. A *hyperplane reflexion* has a hyperplane as its mirror.

A *Coxeter group* is one of the form $G := \langle R_0, \ldots, R_{n-1} \rangle$, the group generated by $R_0, \ldots, R_{n-1}$, which satisfies relations solely of the form

$$(R_i R_j)^{p_{ij}} = E,$$

the identity, where the $p_{ij} = p_{ji}$ are positive integers (or infinity) satisfying $p_{jj} = 1$ for each $j = 0, \ldots, n-1$. In addition, we call G a *string* (Coxeter) group if $p_{ij} = 2$ whenever $0 \leqslant i < j - 1 \leqslant n - 2$; this group is then denoted $[p_1, \ldots, p_{n-1}]$. We shall discuss Coxeter groups in full generality in Chapter 3.

1B1 Theorem *The symmetry group $G(P)$ of a regular convex n-polytope P is a finite string Coxeter group, with generators R_j for $j = 0, \ldots, n-1$ which are hyperplane reflexions, and $p_j := p_{j-1,j} \geqslant 3$ for $j = 1, \ldots, n-1$ (in the previous notation). Conversely, any finite string Coxeter group for which $p_j \geqslant 3$ for $j = 1, \ldots, n-1$ is the symmetry group of a regular convex polytope.*

Proof. Let us explain how this result arises. Fix a flag $\Phi = \{F_{-1}, F_0, \ldots, F_{n-1}, F_n\}$ of a regular n-polytope P, with the conventions introduced previously. Denote by q_j the centroid of F_j for $j = 0, \ldots, n$ (by this, we mean the centroid of its vertices), and, for each $j = 0, \ldots, n-1$, let

1B2 $$H_j := \mathrm{aff}\{q_i \mid i \neq j\}.$$

It is not hard to see that $\{q_0, \ldots, q_n\}$ is affinely independent, so that each H_j is a hyperplane. If R_j is the (hyperplane) reflexion whose mirror is H_j, then $G(P) = \langle R_0, \ldots, R_{n-1} \rangle$.

We see this as follows. In any n-polytope P, and for any flag Φ of P, for each $j = 0, \ldots, n-1$, let Φ^j (as before) be the unique flag of P which is j-adjacent to Φ. Then R_j is the unique symmetry of P which interchanges Φ and Φ^j. The simple-connectedness of the boundary of P (for $n \geqslant 3$ – the case $n = 2$ is trivial) then leads directly to the first assertion of the theorem. Many of the details of the proof are exactly as in that of Theorem 1B3, and so we shall postpone them until then.

We shall leave the converse of Theorem 1B1 until we have discussed Coxeter groups in more detail. However, the essence of the argument lies in the fact that a finite

Coxeter group always admits a faithful representation as a euclidean reflexion group (see Theorem 3B1). □

With the regular n-polytope P, we can associate its *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$, where the p_j are given by Theorem 1B1. We may observe already that the mirrors H_j and their images under $G(P)$ split $\mathbb{E}^n$ up into convex (actually simplicial) cones. The H_j themselves bound one of these, the *fundamental region* for $G(P)$, which is that generated by $q_1, \ldots, q_{n-1}$, with q_0 as apex. For $i \neq j$, the dihedral angle between H_i and H_j is π/p_{ij}. Moreover, for $j = 0, \ldots, n$, we have

$$F_j = \operatorname{conv} q_0 \langle R_0, \ldots, R_{j-1} \rangle,$$

with $F_n = P$ as before. This procedure is known as *Wythoff's construction* (see [120, §11.6; 466]); we shall meet it again in Chapter 5. It exhibits each face F_j as a regular j-polytope.

The Wythoff construction can be applied to q_{n-1}, using the subgroup $\langle R_{n-k}, \ldots, R_{n-1} \rangle$ to give the face $\hat{F}_k$ of a base flag $\{\hat{F}_0, \ldots, \hat{F}_{n-1}\}$ of a dual polytope P^* of P. We see that P^* is also regular; its Schläfli symbol is $\{p_{n-1}, \ldots, p_1\}$, reversing that of P.

The number "3" occurs frequently in Schläfli symbols, and so we adopt a suitable brief notation; more generally, a string $p, \ldots, p$ of length k is abbreviated to p^k. With this convention, we can list all the regular (convex) n-polytopes (see [120, Chapter 7; 167, Chapter I.4, I.5; 217, §23]). They are:

- for $n = 0$, the point;
- for $n = 1$, the (line-) segment: $\{\ \}$;
- for $n = 2$, for each $p \geqslant 3$, a polygon: $\{p\}$;
- for $n = 3$, five polyhedra: $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$, $\{5, 3\}$;
- for $n = 4$, six polytopes: $\{3, 3, 3\}$, $\{3, 3, 4\}$, $\{4, 3, 3\}$, $\{3, 4, 3\}$, $\{3, 3, 5\}$, $\{5, 3, 3\}$;
- for $n \geqslant 5$, three polytopes: $\{3^{n-1}\}$, $\{3^{n-2}, 4\}$, $\{4, 3^{n-2}\}$.

We shall see why this list is complete in Chapter 3 (compare Table 3B1). The regular polyhedra are the five Platonic solids, namely, the tetrahedron, octahedron, cube, icosahedron and dodecahedron, respectively. For $n = 4$, the regular polytopes are the 4-simplex, 4-cross-polytope, 4-cube, 24-cell, 600-cell and 120-cell, respectively (see also [403] for historical comments about the 120-cell). For $n \geqslant 5$ we only have the n-simplex, n-cross-polytope and n-cube, respectively.

The concept of regular convex polytope in $\mathbb{E}^n$ can be generalized in many different ways. In this section, we stay within the context of convex polytopes; in the following sections, we shall generalize the concept in various other directions.

An *automorphism* of a polytope P is a permutation γ of its face-lattice $\mathcal{P}$ which preserves inclusion; that is, γ is an automorphism of $\mathcal{P}$ in the usual sense. The automorphisms of P form a group $\Gamma(P)$, called the *automorphism group* of P. Following [277] (see also [278]), we say that P is *combinatorially regular* if $\Gamma(P)$ is transitive on $\mathcal{F}(P)$. As examples, every n-polytope with $n \leqslant 2$ is combinatorially regular, as is every simplex.

The main result of [277] is

1B3 Theorem *A combinatorially regular polytope is isomorphic to an ordinary regular polytope.*

Proof. We shall sketch the main details of the proof here. The core idea is to show that, if P is combinatorially regular, then $\Gamma(P)$ is a finite Coxeter group. We may then appeal to two facts: first, a finite Coxeter group is always the automorphism group of some regular convex polytope (this uses Wythoff's construction and the fact that a finite Coxeter group is isomorphic to a reflexion group, both mentioned previously); second, two combinatorially regular polytopes are isomorphic if and only if their automorphism groups are isomorphic.

So, with the same conventions as before, we fix a *base* flag $\Phi := \{F_{-1}, F_0, \ldots, F_n\}$ of the combinatorially regular n-polytope P. For each $j = 1, \ldots, n-1$, there is a $p_j \geqslant 3$, such that the section F_{j+1}/F_{j-2} is a p_j-gon; by the combinatorial regularity of P, the same is true of each flag of P, with the same numbers p_j. We shall show that

1B4 $$\Gamma(P) \cong [p_1, \ldots, p_{n-1}],$$

the string Coxeter group defined previously.

If Ψ is any flag of P, then, as before, for $j = 0, \ldots, n-1$ we write Ψ^j for the j-adjacent flag of P to Ψ. It is immediately clear that, if $\gamma \in \Gamma(P)$, then $\Psi^j\gamma = (\Psi\gamma)^j$. From this, we deduce that, if $\gamma \in \Gamma(P)$ is such that $\Phi\gamma = \Phi$, then $\gamma = \varepsilon$, the identity, since γ will also fix each flag adjacent to Φ, and thus all flags of P, using the flag-connectedness of P. As a further consequence, there is a one-to-one correspondence between $\mathcal{F}(P)$ and $\Gamma(P)$, so that $|\Gamma(P)| = \operatorname{card} \mathcal{F}(P)$.

For each $j = 0, \ldots, n-1$, there is thus a unique $\rho_j \in \Gamma(P)$, such that $\Phi\rho_j := \Phi^j$. We claim that $\Gamma(P) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and then establish the required isomorphism. The reason that the ρ_j generate $\Gamma(P)$ is straightforward. If $\gamma \in \Gamma(P)$, write $\Psi := \Phi\gamma$. Since $\mathcal{F}(P)$ is connected, we may find a sequence $\Phi = \Phi_0, \ldots, \Phi_r = \Psi$ of flags, such that Φ_{s-1} and Φ_s are adjacent for each $s = 1, \ldots, r$. Thus for each s, we have $\Phi_s = \Phi_{s-1}^{j(s)}$ for some $j(s) \in \{0, \ldots, n-1\}$. There is now an easy induction argument on r, using the observation we made before, whose general step for $s \geqslant 1$ is

1B5 $$\Phi_s = \Phi_{s-1}^{j(s)} = \left(\Phi\rho_{j(s-1)} \cdots \rho_{j(1)}\right)^{j(s)} = \Phi\rho_{j(s)} \cdots \rho_{j(1)}.$$

With $s = r$, appealing to the uniqueness of γ leads to

1B6 $$\gamma = \rho_{j(r)} \cdots \rho_{j(1)} \in \langle \rho_0, \ldots, \rho_{n-1} \rangle,$$

as we wished to show.

We next need to establish that the relations explicitly given by the definition of $[p_1, \ldots, p_{n-1}]$ in terms of its generators $\rho_0, \ldots, \rho_{n-1}$ suffice to determine $\Gamma(P)$. We proceed as follows. If

$$\rho_{j(r)} \cdots \rho_{j(1)} = \varepsilon$$

is a relation in $\Gamma(P)$ involving certain of its generators, then

$$\Phi^{j(1)\cdots j(r)} = \Phi,$$

by reversing the earlier discussion. We can now show where the various kinds of relations arise; indeed, they come from

$$\Phi^{(ij)^{p_{ij}}} = \Phi,$$

which just describe the structure of various sections of P. (The sections of dimensions 1 and 2 give $p_{jj} = 1$ and the values of the $p_j = p_{j-1,j}$; the other relations arise from $\Phi^{ij} = \Phi^{ji}$ if $i < j - 1$.)

We now look at a general relationship of this kind in a more geometric way. With a flag $\Psi = \{G_0, \ldots, G_{n-1}\}$ (we suppress the improper faces), we associate an $(n-1)$-simplex

1B7 $$T(\Psi) := \operatorname{conv}\{q(G_j) \mid j = 0, \ldots, n-1\},$$

where $q(G)$ denotes the centroid of (the vertices of) the polytope G. (In effect, we are constructing the order complex of the face-lattice of P; see Section 2C; as is often the case, we identify a simplicial complex with its underlying point-set – its "polyhedron".) Two such $(n-1)$-simplices share a common $(n-2)$-face if and only if the corresponding flags are adjacent. We associate with this sequence of adjacent flags an *interior loop* in the boundary $\operatorname{bd} P$ of P; this is a continuous loop in $\operatorname{bd} P$ which passes successively from one simplex $T(\Psi)$ to the next adjacent one, and does not meet an $(n-3)$-face of any of them. If $n \leqslant 2$, there is nothing to be said, since we have already identified such n-polytopes as combinatorially regular (and then their groups $\Gamma(P)$ are as described). When $n \geqslant 3$, we can appeal to the fact that $\operatorname{bd} P$ is simply connected. This means that any loop in $\operatorname{bd} P$ is contractable to a point within $\operatorname{bd} P$. We thus contract the interior loop associated with a relation in $\Gamma(P)$ to a point in $T(\Phi)$; clearly, we can do this in such a way that it never meets a face of any of the simplices $T(\Psi)$ of dimension less than $n-3$. The resulting operations on the sequences of flags are of two types:

- moving to an adjacent flag, and then returning, corresponding to a relation $\Psi^{jj} = \Psi$;
- moving the loop over an $(n-3)$-face of some simplex $T(\Psi)$, corresponding to a relation $\Psi^{(ij)^{p_{ij}}} = \Psi$.

Each gives a relation in $\Gamma(P)$ conjugate to one of those defining the Coxeter group $[p_1, \ldots, p_{n-1}]$; we conclude that $\Gamma(P) \cong [p_1, \ldots, p_{n-1}]$, as was claimed. This ends our discussion of the proof, for reasons that we have already indicated. □

In fact, a careful analysis of this proof, specifically the inductive step in (1B5), shows that something considerably short of the full force of combinatorial regularity will yield the same consequences.

1B8 Corollary *An n-polytope P is combinatorially regular if and only if there exists a flag Φ of P, such that, for each $j = 0, \ldots, n-1$, there is a $\rho_j \in \Gamma(P)$, such that $\Phi^j = \Phi\rho_j$.*

In [279], various other criteria for combinatorial regularity were established. The most important is [279, Theorem 4A1], which was rediscovered in [155]; we introduce it in a more general context here. Call an n-polytope P *equivelar* if, for each $j = 1, \ldots, n-1$, there exists a number p_j, such that, for each flag $\Psi = \{G_{-1}, G_0, \ldots, G_n\}$ of P, the section G_{j+1}/G_{j-2} is a p_j-gon. (The concept of equivelarity was introduced in a somewhat different context in [310] for polyhedral manifolds; see Section 1D.) This is now a purely combinatorial notion; that is, it does not explicitly mention the automorphism group $\Gamma(P)$. Nevertheless, we have

1B9 Theorem *An equivelar convex polytope is combinatorially regular.*

Proof. We may clearly take $n \geqslant 3$ here. The essence of the proof is the following. If Φ is a given flag of P, any other flag is of the form

$$\Phi^{j(1)\cdots j(r)},$$

for some $j(1), \ldots, j(r)$. Given another flag Ψ of P, we define a mapping $\gamma := \gamma_\Psi : \mathcal{F}(P) \to \mathcal{F}(P)$ by

$$\Phi^{j(1)\cdots j(r)}\gamma := \Psi^{j(1)\cdots j(r)}.$$

For a general polytope, this mapping would not be well defined. However, since P is equivelar and bd P is simply connected, any alternative expression

$$\Phi^{k(1)\cdots k(s)} = \Phi^{j(1)\cdots j(r)}$$

leads to an interior loop associated with

$$\Phi^{k(1)\cdots k(s)j(r)\cdots j(1)} = \Phi,$$

and reversing the contraction of the loop, but now basing it on Ψ, leads to

$$\Psi^{k(1)\cdots k(s)j(r)\cdots j(1)} = \Psi,$$

or

$$\Psi^{k(1)\cdots k(s)} = \Psi^{j(1)\cdots j(r)},$$

showing that γ is well defined. We have therefore explicitly constructed $\gamma \in \Gamma(P)$ such that $\Phi\gamma = \Psi$; hence P is combinatorially regular. □

Various other criteria for combinatorial regularity of a convex n-polytope P can be deduced from Theorem 1B9. These sit between the original definition and its reformulation in terms of equivelarity; they are all taken from [279], and we list them without proof.

- $\Gamma(P)$ is transitive on the facets of P, one facet F of P is combinatorially regular, and its automorphism group $\Gamma(F)$ is a subgroup of $\Gamma(P)$. (There is an obvious dual criterion in terms of vertex-figures.)
- For each $j = 1, \ldots, n-1$, the group $\Gamma(P)$ is transitive on the incident pairs of $(j-2)$- and $(j+1)$-faces of P.

The last two criteria involve a common condition on a polytope P.

1B10 For each face F of P, and each $\gamma \in \Gamma(P)$ such that $F\gamma = F$, there exist (commuting) $\gamma_-, \gamma_+ \in \Gamma(P)$, such that $\gamma = \gamma_-\gamma_+$, and γ_- (γ_+) fixes each face G of P with $G \subset F$ ($F \subset G$).

- $\Gamma(P)$ satisfies (1B10), and, for each $j = 1, \ldots, n-1$, is transitive on the incident pairs of $(j-1)$- and j-faces of P.
- $\Gamma(P)$ satisfies (1B10), and there exists a flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of P such that, for each $j = 0, \ldots, n-1$, there is a $\sigma_j \in \Gamma(P)$ which interchanges F_j and the unique other j-face F_j^* of P which satisfies $F_{j-1} \subset F_j^* \subset F_{j+1}$.

The very stringent requirements in the definition of regularity of polytopes can be relaxed in many different ways, yielding a great variety of weaker "regularity" notions. For example, a convex polytope is called *semi-regular* if its facets are regular and its symmetry group is transitive on the vertices (see [32, 103, 108, 123, 124, 235]). There are also other related concepts, but we shall not employ any of them; for a survey, see [273, 371].

1C Extensions of Regularity

We now treat some different notions of regular polytope; we shall gradually increase the level of generality. First, there are the infinite analogues in ordinary euclidean space, the regular honeycombs. A *honeycomb* or *tessellation* is a collection P of n-polytopes, called *cells*, which tiles $\mathbb{E}^n$ face-to-face; that is, two of its cells have disjoint interiors and meet on a common face of each (which may be empty), and these cells cover $\mathbb{E}^n$. A *flag* of a honeycomb P is defined as for an $(n+1)$-polytope, of which it is an infinite analogue; then F_n will be a cell of P, which is why we also use the alternative designation *facet*.

As might be expected, a honeycomb P will be regular if its symmetry group $G(P)$ (the group of isometries of $\mathbb{E}^n$ which preserves P) is transitive on the flags of P. Once again, $G(P)$ will be a Coxeter group generated by hyperplane reflexions; for $j = 0, \ldots, n$, the mirror H_j of the jth generating reflexion R_j is defined exactly as (1B2) (there is no centre q_{n+1}, of course). This time, though, the fundamental region for $G(P)$ is an n-simplex $\text{conv}\{q_0, \ldots, q_n\}$.

The regular honeycombs in $\mathbb{E}^n$ can be classified; again, we refer forward to Chapter 3 for details (see Table 3B2). The list is

- for $n = 1$, the single *apeirogon*: $\{\infty\}$;
- for $n = 2$, three tessellations: $\{4, 4\}$, $\{3, 6\}$, $\{6, 3\}$;
- for $n = 3$, a single honeycomb: $\{4, 3, 4\}$;
- for $n = 4$, three honeycombs: $\{4, 3, 3, 4\}$; $\{3, 3, 4, 3\}$, $\{3, 4, 3, 3\}$;
- for $n \geqslant 5$, a single honeycomb: $\{4, 3^{n-2}, 4\}$.

For each n, we have the tessellation by cubes; this is the apeirogon if $n = 1$, or $\{4, 3^{n-2}, 4\}$ if $n \geqslant 2$, giving the square tessellation $\{4, 4\}$ if $n = 2$. In the plane, there are also the triangular and hexagonal tessellations $\{3, 6\}$ and $\{6, 3\}$, respectively. Finally, in dimension 4, we also have the two exceptional tessellations $\{3, 3, 4, 3\}$ and $\{3, 4, 3, 3\}$, with 4-cross-polytopes or 24-cells as tiles, respectively.

Having permitted honeycombs in euclidean space, the next natural step is to generalize yet further to hyperbolic space. Again, we shall consider such honeycombs in detail in Chapter 3, where a complete classification of them will be given (see Table 3C1).

Next, one can allow non-convex polytopes, such as pentagrams and their higher dimensional analogues. At first, we proceed in an elementary way, constructing such polytopes by building them up, facet by facet. We begin with the planar regular polygons; non-planar polygons will only be talked about a little later. A *polygon* (more exactly, a *p-gon*) is now defined to be a finite set of line-segments $[a_{i-1}a_i]$ $(i = 1, \ldots, p)$ in a (2-dimensional) plane, with $a_p = a_0$. Thus a polygon is thought of as a closed path formed by line-segments which are its *edges* (facets); the end points of these segments are its *vertices*.

As before, a polygon P has a symmetry group $G(P)$, and P is regular if $G(P)$ is transitive on the set $\mathcal{F}(P)$ of flags of P. Since $G(P)$ permutes the vertices of P, their centroid

$$c := \frac{1}{p} \sum_{i=0}^{p-1} a_i$$

is invariant under $G(P)$. If we choose c to be the origin of the coordinate system in the plane, then $G(P)$ will be an orthogonal group. The vertices a_i will lie on a circle centred at c, the edges of P will have the same length, and the angles of P at its vertices will be the same. Thus we have a less abstract concept of regularity than in the formal definition.

An edge $[a_{i-1}a_i]$ of P will subtend an angle $2d\pi/p$ at c, for some integer d satisfying $1 \leqslant d < \frac{1}{2}p$, with the greatest common divisor $(p, d) = 1$. If $d > 1$, then P will be non-convex, and is called a *regular star-polygon*. The Schläfli symbol for P is then $\{\frac{p}{d}\}$.

For $n \geqslant 3$, we define a *regular star-polytope* P as follows. This will be formed from finitely many $(n-1)$-polytopes, meeting in pairs on their $(n-2)$-faces. These facets are congruent and regular, and the vertex-figures of P (which must exist) are also congruent regular $(n-1)$-polytopes. One can also make a more abstract definition, regarding P as a configuration of affine subspaces of various dimensions from 0 to $n-1$ (these substitute for the faces of the same dimensions) satisfying the same conditions as those we listed for a convex polytope; the symmetry group $G(P)$ will be obliged to be transitive on the flags of P.

Whichever definition we adopt, we conclude, much as in the convex case, that $G(P)$ is a finite group generated by hyperplane reflexions. From this point, one may proceed in one of several ways; we briefly trace the historical development. As we remarked in Section 1A, the regular pentagram $\{\frac{5}{2}\}$ was already known to the Greeks, and the regular star-polygons were systematically investigated by Bradwardine. The completeness of the enumeration of the regular star-polyhedra (Kepler–Poinsot polyhedra) was established by Cauchy, and the remaining regular star-polytopes (which are confined to $\mathbb{E}^4$) were found by Schläfli and Hess.

These constructions were all more or less recursive, and hence synthetic. It really had to await Coxeter's interest in them for group theory to begin to play the pre-eminent role it does now. Coxeter also laid the foundations for a number of systematic constructions which relate different regular polytopes. Thus, for example, while Hess had shown, by enumeration of all the cases, that a regular star-polytope must have the same vertices as those of some regular convex polytope, a complete explanation of this result was not provided until [280, p. 592]. We shall discuss this result in an appropriate context in Section 7D.

A further generalization of the notion of regular polyhedron arose from an observation by Petrie in 1926. One still insists on convex polygonal faces, but now allows the vertex-figures to be regular skew (non-planar) polygons. He and Coxeter did not take the next step of allowing skew faces as well; this was done by Grünbaum in 1965. Finally, there are abstract generalizations, such as regular maps, which we shall begin to discuss in Section 1D. Each concept leads to a greater level of generalization. The definitions here will anticipate those in Chapters 2 and 5, to which reference should be made.

1D Regular Maps

It is no part of our overall plan to give the whole theory of regular maps at this stage. What we wish to do, rather, is to describe a number of examples which we can then use to illustrate the next few chapters.

Informally, a *map* $\mathcal{P}$ is a family of polygons (which for our present purposes may be apeirogons, that is, infinite), such that

- any two polygons meet in a common edge or vertex, or do not meet at all;
- each edge belongs to precisely two polygons;
- the polygons containing a given vertex form a single cycle of adjacent polygons (sharing a common edge);
- between any two polygons is a chain of adjacent polygons.

The map $\mathcal{P}$ will have an automorphism group $\Gamma(\mathcal{P})$, and $\mathcal{P}$ will be *regular* if $\Gamma(\mathcal{P})$ is transitive on the set $\mathcal{F}(\mathcal{P})$ of flags of $\mathcal{P}$. If the regular map $\mathcal{P}$ has p-gonal faces and q-gonal vertex-figures, then it is said to be of (*Schläfli*) *type* $\{p, q\}$.

Each regular polyhedron or apeirohedron gives rise to a regular map, in the natural way. We wish to describe several classes of such regular maps, which admit more or less concrete definitions; these are in terms of various edge-paths on the map.

First among these is the *Petrie property*. Locally, at least, a map has a definite orientation, relative to one of its polygons, say. (Globally, a map need not be orientable, even if it is finite. If a map has infinitely many polygons through each vertex, it is often hard to discuss its underlying topology in any sensible way.) An edge-path which uses two successive edges of a polygonal face, but not three, is called a *Petrie polygon* of $\mathcal{P}$. (This notion is traditionally attributed to J. F. Petrie; but only recently did we learn from Professor Coxeter that such polygons were investigated before Petrie in Reinhardt [345, p. 11], where the concept was introduced for general convex polyhedra. However, it seems that its significance for regular polytopes was indeed first pointed out by Petrie.) All Petrie polygons are equivalent under $\Gamma(\mathcal{P})$. A regular map of type $\{p, q\}$ which is determined by the length r of its Petrie polygons is denoted by $\{p, q\}_r$. A famous example of such a regular map is Klein's map $\{3, 7\}_8$, connected with the solution of polynomial equations of degree 7 (see [250, p. 461; 251, p. 260; 252, p. 109]); another is Dyck's map $\{3, 8\}_6$ (see [157, p. 488; 158]).

The path in a regular map $\mathcal{P}$ formed by the edges which successively take the second exit on the left (in a local orientation), rather than the first, at each vertex, is called a *hole*. Again, all (left and right) holes of $\mathcal{P}$ are equivalent under $\Gamma(\mathcal{P})$. A regular map of type $\{p, q\}$ which is determined by the length h of its holes is denoted by $\{p, q \mid h\}$. We give just a few examples here. The Poinsot polyhedron $\{5, \frac{5}{2}\}$ is isomorphic to $\{5, 5 \mid 3\}$. The map $\{4, 4 \mid h\}$, for $h \geqslant 3$, is obtained from an $h \times h$ array of squares by directly identifying opposite sides to form a torus (later, we shall permit the case $h = 2$ as well). Finally, the Petrie–Coxeter regular skew polyhedra mentioned in Section 1C are $\{4, 6 \mid 4\}$ and its dual $\{6, 4 \mid 4\}$, discovered by Petrie, and $\{6, 6 \mid 3\}$, found immediately afterwards by Coxeter.

The last class we wish to consider at this stage comprises the regular *toroidal polyhedra* or, more briefly, the *regular tori*. The symmetry groups of the three planar regular tessellations or tilings $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$, by triangles, squares and hexagons, respectively, contain (normal) subgroups of translations. For the first two, the subgroups are transitive on the vertices of the tilings, and for the last two, they are transitive on the faces. We may then identify vertices, edges and faces of such a tiling $\mathcal{P}$ by a subgroup T of the translations. If T is a normal subgroup of the whole symmetry group, then (with a few exceptions) the map $\mathcal{P}/T$ obtained by this identification will be regular. The exceptional cases arise when T is too large a subgroup of the whole translation group.

Such a regular map is alternatively designated by a symbol $\mathcal{P}_{\mathbf{s}}$, where $\mathbf{s} := (s, t)$ is a certain non-negative integer vector. Let us begin with the easiest case $\mathcal{P} = \{4, 4\}$. We can take the vertices of $\{4, 4\}$ to be those points of $\mathbb{E}^2$ with integer cartesian coordinates. If (s, t) is any non-zero integer vector, then (s, t) and the point $(t, -s)$ obtained by rotating it by $\pi/2$ about the origin $o := (0, 0)$ generate a subgroup $T = \Lambda_{(s,t)}$ of translations (by integer linear combinations). The map resulting from $\{4, 4\}$ by identification under $\Lambda_{(s,t)}$ is denoted by $\{4, 4\}_{\mathbf{s}} = \{4, 4\}_{(s,t)} := \{4, 4\}/\Lambda_{(s,t)}$. This map will have the topological type of a torus, whence the general descriptive name.

Now, as we have defined it, the lattice $\Lambda_{(s,t)}$ has the full rotational symmetry of the whole tiling $\{4, 4\}$. This has two consequences. First, we may replace (s, t), if necessary, by one of its images under rotation by multiples of $\pi/2$, to make $s,\ t \geqslant 0$; this is unique, except when one of the coordinates is zero, in which case we take $(s, 0)$ in preference to $(0, s)$. Second, the identified map $\{4, 4\}_{(s,t)}$ will also have full rotational symmetry. However, it will only have full reflexional symmetry, so that it is regular, if $(t, s) \in \Lambda_{(s,t)}$ also; bearing in mind that (s, t) and its rotational images generate $\Lambda_{(s,t)}$, this means that $t = 0$ or $t = s$. Restoring the symmetry between s and t enables us to write this condition in the form $st(s - t) = 0$. If $st(s - t) \neq 0$, then $\{4, 4\}_{(s,t)}$ will only have rotational symmetry; the technical term to describe such a map is *chiral*,

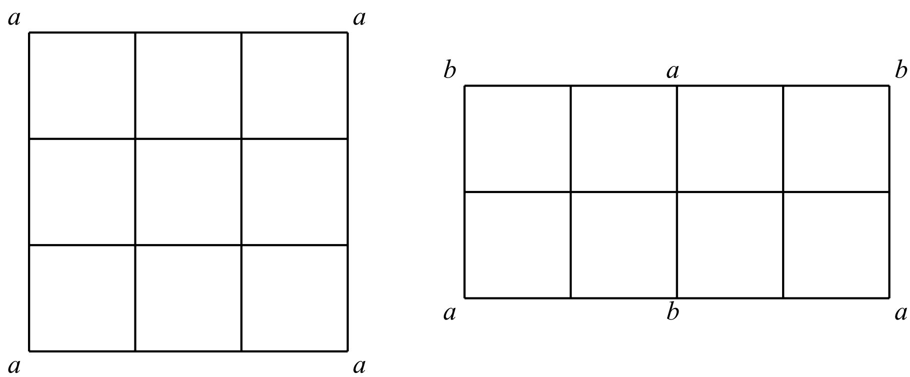

Figure 1D1. The tori $\{4, 4\}_{(3,0)}$ and $\{4, 4\}_{(2,2)}$.

which will be defined generally at the end of Section 2B. Incidentally, the "small" vectors $(s, t) = (1, 0)$ and $(1, 1)$ (which generate "large" translation subgroups) must be excluded here, because the resulting maps degenerate. Two small examples, namely, $\{4, 4\}_{(3,0)}$ and $\{4, 4\}_{(2,2)}$, are illustrated in Figure 1D1; points with the same label are identified by translation.

We may treat the other two tilings $\{3, 6\}$ and $\{6, 3\}$ similarly, and, in fact, together. The vertices of $\{3, 6\}$ (or the centres of the faces of $\{6, 3\}$) may also be taken to be the points with integer coordinates, but this time with respect to an oblique pair of axes, at an angle $\pi/3$. Again, a non-zero integer vector (s, t) generates a translation subgroup $\Lambda_{(s,t)}$, with the aid of its images by rotations through multiples of $\pi/3$, namely, $(t, -s)$ and $(t - s, s + t)$ (we may ignore their negatives). Thus, we may suppose that (s, t) is a non-negative vector. As before, we have maps $\{3, 6\}_{\mathbf{s}} = \{3, 6\}_{(s,t)}$ and $\{6, 3\}_{\mathbf{s}} = \{6, 3\}_{(s,t)}$ on the torus. They will generally be chiral: the condition for regularity is again $st(s - t) = 0$. Observe that, in this case, only the "small" vector $(s, t) = (1, 0)$ must be excluded. As with the other tori, two small examples, namely, $\{3, 6\}_{(3,0)}$ and $\{6, 3\}_{(1,1)}$, illustrate this in Figure 1D2.

These notions can be generalized in many ways, but we shall postpone further discussion until we can do it in terms of the groups. However, let us briefly mention

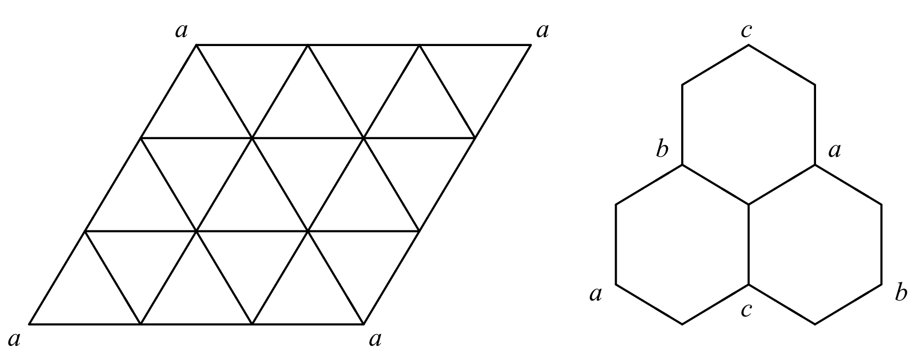

Figure 1D2. The tori $\{3, 6\}_{(3,0)}$ and $\{6, 3\}_{(1,1)}$.

(if only to dismiss) a generalization of regularity for maps formulated in [310]. A map is called *equivelar* of *type* $\{p, q\}$ if each of its faces is a p-gon, and these faces meet q at each vertex. A regular map must, of course, be equivelar, but the converse is far from true. Large classes of equivelar maps were constructed in [310, 311], where the interest was in whether they admitted polyhedral embeddings in $\mathbb{E}^3$; some of these maps are regular, particularly the simplest cases of those of type $\{4, q\}$ and $\{p, 4\}$ (see [50, 51, 309]).