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Nonlinear Elasticity: Theory and Applications

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Elements of the theory of finite elasticity

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In this chapter we provide a brief overview of the main ingredients of the nonlinear theory of elasticity in order to establish the basic background material as a reference source for the other, more specialized, chapters in this volume.

1.1 Introduction

In this introductory chapter we summarize the basic equations of nonlinear elasticity theory as a point of departure and as a reference source for the other articles in this volume which are concerned with more specific topics.

There are several texts and monographs which deal with the subject of nonlinear elasticity in some detail and from different standpoints. The most important of these are, in chronological order of the publication of the first edition, Green and Zerna (1954, 1968, 1992), Green and Adkins (1960, 1970), Truesdell and Noll (1965), Wang and Truesdell (1973), Chadwick (1976, 1999), Marsden and Hughes (1983, 1994), Ogden (1984a, 1997), Ciarlet (1988) and Antman (1995). See also the textbook by Holzapfel (2000), which deals with viscoelasticity and other aspects of nonlinear solid mechanics as well as containing an extensive treatment of nonlinear elasticity. These books may be referred to for more detailed study. Subsequently in this chapter we shall refer to the most recent editions of these works. The review articles by Spencer (1970) and Beatty (1987) are also valuable sources of reference.

Section 1.2 of this chapter is concerned with laying down the basic equations of elastostatics and it includes a summary of the relevant geometry of deformation and strain, an account of stress and stress tensors, the equilibrium equations and boundary conditions and an introduction to the formulation of constitutive laws for elastic materials, with discussion of the important notions of objectivity and material symmetry. Some emphasis is placed on the special case of isotropic elastic materials, and the constitutive laws for anisotropic material consisting of one or two families of fibres are also discussed. The modifications to the constitutive laws when internal constraints such as incompressibility and inextensibility are present are provided. The general boundary-value problem of nonlinear elasticity is then formulated and the circumstances when this can be cast in a variational structure are discussed briefly.

In Section 1.3 some basic examples of boundary-value problems are given. Specifically, the equations governing some homogeneous deformations are highlighted, with the emphasis on incompressible materials. Other chapters in this volume will discuss a range of different boundary-value problems involving non-homogeneous deformations so here we focus attention on just one problem as an exemplar. This is the problem of extension and inflation of a thick-walled circular cylindrical tube. The analysis is given for an incompressible isotropic elastic solid and also for a material with two mechanically equivalent symmetrically disposed families of fibres in order to illustrate some differences between isotropic and anisotropic response.

The (linearized) equations of incremental elasticity associated with small deformations superimposed on a finite deformation are summarized in Section 1.4. The incremental constitutive law for an elastic material is used to identify the (fourth-order) tensor of elastic moduli associated with the stress and deformation variables used in the formulation of the governing equations, and explicit expressions for the components of this tensor are given in the case of an isotropic material. For the two-dimensional specialization, necessary and sufficient conditions on these components for the strong ellipticity inequalities to hold are given for both unconstrained and incompressible materials. A brief discussion of incremental uniqueness and stability is then given in the context of the dead-load boundary-value problem and the associated local inequalities are given explicit form for an isotropic material, again for both unconstrained and incompressible materials. A short discussion of global aspects of non-uniqueness for an isotropic material sets the incremental results in a broader context.

In Section 1.5 the equations of incremental deformations and equilibrium given in Section 1.4 are specialized to the plane strain context in order to provide a formulation for the analysis of incremental plane strain boundary-value problems. Specifically, we provide an example of a typical incremental boundary-value problem by considering bifurcation of a uniformly deformed half-space from a homogeneously deformed configuration into a non-homogeneous local mode of deformation. An explicit bifurcation condition is given for this problem and the results are illustrated for two forms of strain-energy function.

Finally, in Section 1.6 we summarize the equations associated with the (nonlinear) dynamics of an elastic body at finite strain. The (linearized) equations for small motions superimposed on a static finite deformation are then given and these are applied to the analysis of plane waves propagating in a homogeneously deformed material.

References are given throughout the text but these are not intended to provide an exhaustive list of original sources. Where appropriate we mention papers and books where more detailed citations can be found. Also, where a topic is to be dealt with in detail in one of the other chapters of this volume the appropriate citations are included there.

1.2 Elastostatics

In this section we summarize the basic equations of the static theory of nonlinear elasticity, including the kinematics of deformation, the analysis of stress and the governing equations of equilibrium, and we introduce the various forms of constitutive law for an elastic material, including a discussion of isotropy and anisotropy. We then formulate the basic boundary-value problem of nonlinear elasticity. The development here is a synthesis of the essential material taken from the book by Ogden (1997) with some minor differences and additions.

1.2.1 Deformation and strain

We consider a continuous body which occupies a connected open subset of a three-dimensional Euclidean point space, and we refer to such a subset as a *configuration* of the body. We identify an arbitrary configuration as a *reference configuration* and denote this by $\mathcal{B}_r$. Let points in $\mathcal{B}_r$ be labelled by their position vectors $\mathbf{X}$ relative to an arbitrarily chosen origin and let $\partial\mathcal{B}_r$ denote the boundary of $\mathcal{B}_r$. Now suppose that the body is deformed quasi-statically from $\mathcal{B}_r$ so that it occupies a new configuration, $\mathcal{B}$ say, with boundary $\partial\mathcal{B}$. We refer to $\mathcal{B}$ as the *current* or *deformed configuration* of the body. The deformation is represented by the mapping $\chi : \mathcal{B}_r \to \mathcal{B}$ which takes points $\mathbf{X}$ in $\mathcal{B}_r$ to points $\mathbf{x}$ in $\mathcal{B}$. Thus,

$$\mathbf{x} = \chi(\mathbf{X}), \qquad \mathbf{X} \in \mathcal{B}_r, \tag{2.1}$$

where $\mathbf{x}$ is the position vector of the point $\mathbf{X}$ in $\mathcal{B}$. The mapping χ is called the *deformation* from $\mathcal{B}_r$ to $\mathcal{B}$. We require χ to be one-to-one and we write its inverse as χ^{-1}, so that

$$\mathbf{X} = \chi^{-1}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}. \tag{2.2}$$

Both χ and its inverse are assumed to satisfy appropriate regularity conditions. Here, it suffices to take χ to be twice continuously differentiable, but different requirements may be specified in other chapters of this volume.

For simplicity we consider only Cartesian coordinate systems and let $\mathbf{X}$ and $\mathbf{x}$ respectively have coordinates X_α and x_i, where $\alpha, i \in \{1,2,3\}$, so that $x_i = \chi_i(X_\alpha)$. Greek and Roman indices refer, respectively, to $\mathcal{B}_r$ and $\mathcal{B}$ and the usual summation convention for repeated indices is used.

The *deformation gradient tensor*, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} \tag{2.3}$$

and has Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, Grad being the gradient operator in $\mathcal{B}_r$. Local invertibility of χ requires that $\mathbf{F}$ be non-singular, and we adopt the usual convention that $\det \mathbf{F} > 0$. Similarly, for the inverse deformation gradient

$$\mathbf{F}^{-1} = \mathrm{grad}\,\mathbf{X}, \quad (\mathbf{F}^{-1})_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \tag{2.4}$$

where grad is the gradient operator in $\mathcal{B}$. With use of the notation defined by

$$J = \det \mathbf{F} \tag{2.5}$$

we then have

$$0 < J < \infty. \tag{2.6}$$

The equation

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X} \tag{2.7}$$

(in components $\mathrm{d}x_i = F_{i\alpha}\mathrm{d}X_\alpha$) describes how an infinitesimal *line element* $\mathrm{d}\mathbf{X}$ of material at the point $\mathbf{X}$ transforms *linearly* under the deformation into the line element $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$.

We now set down how elements of surface area and volume transform. Let $\mathrm{d}\mathbf{A} \equiv \mathbf{N}\mathrm{d}A$ denote a vector surface area element on $\partial\mathcal{B}_r$, where $\mathbf{N}$ is the unit outward normal to the surface, and $\mathrm{d}\mathbf{a} \equiv \mathbf{n}\mathrm{d}a$ the corresponding area element on $\partial\mathcal{B}$. Then, the area elements are connected according to *Nanson's formula*

$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A, \tag{2.8}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$ and ^{T} denotes the transpose. Note that, unlike a line element, the normal vector is not embedded in the material, i.e. $\mathbf{n}$ is not in general aligned with the same line element of material as $\mathbf{N}$.

If $\mathrm{d}V$ and $\mathrm{d}v$ denote volume elements in $\mathcal{B}_r$ and $\mathcal{B}$ respectively then we also have

$$\mathrm{d}v = J\mathrm{d}V. \tag{2.9}$$

For a volume preserving (*isochoric*) deformation we have

$$J = \det \mathbf{F} = 1. \tag{2.10}$$

A material for which (2.10) is constrained to be satisfied for all deformation gradients $\mathbf{F}$ is said to be *incompressible.*

The identities

$$\mathrm{Div}\,(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \mathrm{div}\,(J^{-1}\mathbf{F}) = \mathbf{0} \tag{2.11}$$

are important tools in transformations between equations associated with the reference and current configurations, where Div and div are the divergence operators in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. The first identity in (2.11) can readily be established by integrating (2.8) over an arbitrary closed surface in $\mathcal{B}$ and applying the divergence theorem and the second similarly by integrating $\mathbf{N}\mathrm{d}A$ over an arbitrary closed surface in $\mathcal{B}_r$.

From (2.7) we have

$$|\mathrm{d}\mathbf{x}|^2 = (\mathbf{F}\mathbf{M}) \cdot (\mathbf{F}\mathbf{M})\,|\mathrm{d}\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot \mathbf{M}\,|\mathrm{d}\mathbf{X}|^2, \tag{2.12}$$

where we have introduced the unit vector $\mathbf{M}$ in the direction of $\mathrm{d}\mathbf{X}$ and $\cdot$ signifies the scalar product of two vectors. Then, the ratio $|\mathrm{d}\mathbf{x}|/|\mathrm{d}\mathbf{X}|$ of the lengths of a line element in the deformed and reference configurations is given by

$$\frac{|\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{X}|} = |\mathbf{F}\mathbf{M}| = [\mathbf{M} \cdot (\mathbf{F}^T\mathbf{F}\mathbf{M})]^{1/2} \equiv \lambda(\mathbf{M}). \tag{2.13}$$

Equation (2.13) defines the *stretch* $\lambda(\mathbf{M})$ *in the direction* $\mathbf{M}$ at $\mathbf{X}$, and we note that it is restricted according to the inequalities

$$0 < \lambda(\mathbf{M}) < \infty. \tag{2.14}$$

If there is no stretch in the direction $\mathbf{M}$ then $\lambda(\mathbf{M}) = 1$ and hence

$$(\mathbf{F}^T\mathbf{F}\mathbf{M}) \cdot \mathbf{M} = 1. \tag{2.15}$$

If there is no stretch in any direction, i.e. (2.15) holds for all $\mathbf{M}$, then the material is said to be *unstrained* at $\mathbf{X}$, and it follows that $\mathbf{F}^T\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor. A suitable tensor measure of strain is therefore $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ since this tensor vanishes when the material is unstrained. This leads to the definition of the *Green strain tensor*

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \tag{2.16}$$

where the $1/2$ is a normalization factor. If, for a given $\mathbf{M}$, equation (2.15) holds for all possible deformation gradients $\mathbf{F}$ then the considered material is said to be *inextensible* in the direction $\mathbf{M}$.

The deformation gradient can be decomposed according to the *polar decompositions*

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.17}$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$, $\mathbf{V}$ are positive definite and symmetric tensors. Each of the decompositions in (2.17) is unique. Respectively, $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors.*

These stretch tensors can also be put in spectral form. For $\mathbf{U}$ we have the *spectral decomposition*

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.18}$$

where $\lambda_i > 0$, $i \in \{1,2,3\}$, are the *principal stretches*, $\mathbf{u}^{(i)}$, the (unit) eigenvectors of $\mathbf{U}$, are called the *Lagrangian principal axes* and $\otimes$ denotes the tensor product. Note that $\lambda(\mathbf{u}^{(i)}) = \lambda_i$ in accordance with the definition (2.13). Similarly, $\mathbf{V}$ has the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.19}$$

where

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}, \quad i \in \{1,2,3\}. \tag{2.20}$$

It follows from (2.5), (2.17) and (2.18) that

$$J = \lambda_1 \lambda_2 \lambda_3. \tag{2.21}$$

Using the polar decompositions (2.17) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \tag{2.22}$$

These define $\mathbf{C}$ and $\mathbf{B}$, which are called, respectively, the *right* and *left Cauchy-Green deformation tensors.*

More general classes of strain tensors, i.e. tensors which vanish when there is no strain, can be constructed on the basis that $\mathbf{U} = \mathbf{I}$ when the material is unstrained. Thus, for example, we define Lagrangian strain tensors

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \quad m \neq 0, \tag{2.23}$$

$$\mathbf{E}^{(0)} = \ln \mathbf{U}, \quad m = 0, \tag{2.24}$$

where m is a real number (not necessarily an integer). Eulerian strain tensors based on the use of $\mathbf{V}$ may be constructed similarly. See, for example, Doyle and Ericksen (1956), Seth (1964) and Hill (1968, 1970, 1978). Note that for $m = 2$ equation (2.23) reduces to the Green strain tensor (2.16). For discussion of the logarithmic strain tensor (2.24) we refer to, for example, Hoger (1987).

Let ρ_r and ρ be the *mass densities* in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. Then, since the material in the volume element $\mathrm{d}V$ is the same as that in $\mathrm{d}v$ the mass is conserved, i.e. $\rho\mathrm{d}v = \rho_r\mathrm{d}V$, and hence, from (2.9), we may express the *mass conservation equation* in the form

$$\rho_r = \rho J. \tag{2.25}$$

1.2.2 Stress tensors and equilibrium equations

The surface force per unit area (or *stress vector*) on the vector area element $\mathrm{d}\mathbf{a}$ is denoted by $\mathbf{t}$. It depends on $\mathbf{n}$ according to the formula

$$\mathbf{t} = \boldsymbol{\sigma}^T\mathbf{n}, \tag{2.26}$$

where $\boldsymbol{\sigma}$, a second-order tensor independent of $\mathbf{n}$, is called the *Cauchy stress tensor*.

By means of (2.8) the force on $\mathrm{d}a$ may be written as

$$\mathbf{t}\mathrm{d}a = \mathbf{S}^T\mathbf{N}\mathrm{d}A, \tag{2.27}$$

where the *nominal stress tensor* $\mathbf{S}$ is related to $\boldsymbol{\sigma}$ by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.28}$$

The *first Piola-Kirchhoff stress tensor*, denoted here by $\boldsymbol{\pi}$, is given by $\boldsymbol{\pi} = \mathbf{S}^T$ and this will be used in preference to $\mathbf{S}$ in some parts of this volume.

Let $\mathbf{b}$ denote the body force per unit mass. Then, in integral form, the *equilibrium equation* for the body may be written with reference either to $\mathcal{B}$ or $\mathcal{B}_r$. Thus,

$$\int_{\mathcal{B}} \rho\mathbf{b}\mathrm{d}v + \int_{\partial\mathcal{B}} \boldsymbol{\sigma}^T\mathbf{n}\mathrm{d}a = \int_{\mathcal{B}_r} \rho_r\mathbf{b}\mathrm{d}V + \int_{\partial\mathcal{B}_r} \mathbf{S}^T\mathbf{N}\mathrm{d}A = \mathbf{0}. \tag{2.29}$$

On use of the divergence theorem equations (2.29) yield the equivalent equilibrium equations

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \mathbf{0}, \tag{2.30}$$

$$\operatorname{Div}\mathbf{S} + \rho_r\mathbf{b} = \mathbf{0}, \tag{2.31}$$

where again div and Div denote the divergence operators in $\mathcal{B}$ and $\mathcal{B}_r$ respectively. The derivation of the pointwise equations (2.30) and (2.31) requires that the left-hand sides of these equations are continuous (in $\mathcal{B}$ and $\mathcal{B}_r$ respectively). Note that on use of (2.11) and (2.25) equation (2.31) may be converted immediately to (2.30). In components, (2.31) has the form

$$\frac{\partial S_{\alpha i}}{\partial X_\alpha} + \rho_r b_i = 0, \tag{2.32}$$

and similarly for (2.30), where $S_{\alpha i}$ are the components of $\mathbf{S}$ and b_i those of $\mathbf{b}$.

Balance of the moments of the forces acting on the body yields simply $\boldsymbol{\sigma}^T = \boldsymbol{\sigma}$, which may also be expressed as

$$\mathbf{S}^T\mathbf{F}^T = \mathbf{FS}. \tag{2.33}$$

The Lagrangian formulation based on the use of $\mathbf{S}$ and equation (2.31), with $\mathbf{X}$ as the independent variable, is normally preferred in nonlinear elasticity to the Eulerian formulation based on use of $\boldsymbol{\sigma}$ and equation (2.30) with $\mathbf{x}$ as the independent variable since the initial geometry is known, whereas $\mathbf{x}$ depends on the deformation to be determined.

We now consider the work done by the surface and body forces in a virtual displacement $\delta\mathbf{x}$ from the current configuration $\mathcal{B}$. By using the divergence theorem and equation (2.31) we obtain the *virtual work* equation

$$\int_{\mathcal{B}_r} \rho_r \mathbf{b}\cdot\delta\mathbf{x}\,\mathrm{d}V + \int_{\partial\mathcal{B}_r} (\mathbf{S}^T\mathbf{N})\cdot\delta\mathbf{x}\,\mathrm{d}A = \int_{\mathcal{B}_r} \mathrm{tr}\,(\mathbf{S}\delta\mathbf{F})\,\mathrm{d}V, \tag{2.34}$$

where the left-hand side of (2.34) represents the virtual work of the body and surface forces and in the integrand on the right-hand side tr denotes the trace of a second-order tensor and $\delta\mathbf{F} = \mathrm{Grad}\,\delta\mathbf{x}$. The term on the right-hand side is the virtual work of the stresses in the bulk of the material. For a conservative system this latter work is recoverable and is stored as elastic strain energy (this will be discussed in Section 1.2.5.1) but in general it includes a dissipative part. In either case the integrand, which represents the virtual work increment per unit volume in $\mathcal{B}_r$, may be expressed in many alternative forms using different deformation and strain measures.

For example, using (2.16), (2.17) and the symmetry (2.33), we obtain

$$\mathrm{tr}\,(\mathbf{S}\delta\mathbf{F}) = \mathrm{tr}\,(\mathbf{T}^{(1)}\delta\mathbf{U}) = \mathrm{tr}\,(\mathbf{T}^{(2)}\delta\mathbf{E}), \tag{2.35}$$

in which we have defined the *Biot stress tensor* $\mathbf{T}^{(1)}$ (Biot, 1965) and the *second Piola-Kirchhoff stress tensor* $\mathbf{T}^{(2)}$ (both symmetric) by

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{SR} + \mathbf{R}^T\mathbf{S}^T), \quad \mathbf{T}^{(2)} = \mathbf{SF}^{-T} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{2.36}$$

We note the connection

$$\mathbf{T}^{(1)} = \frac{1}{2}(\mathbf{T}^{(2)}\mathbf{U} + \mathbf{U}\mathbf{T}^{(2)}). \tag{2.37}$$

More generally, the expression in (2.35) may be written in terms of the strain tensors $\mathbf{E}^{(m)}$ given by (2.23) and (2.24) and their (symmetric) *conjugate stress tensors* $\mathbf{T}^{(m)}$ as

$$\mathrm{tr}\,(\mathbf{T}^{(m)}\delta\mathbf{E}^{(m)}). \tag{2.38}$$

Note that the examples $m = 1$ and $m = 2$ from (2.35) are included in (2.38) as special cases. The notion of conjugate stress and strain tensors was introduced by Hill (1968) and applies more generally than to the special class of strain tensors (2.23). A more detailed discussion can be found in Ogden (1997). We observe that the definition of conjugate stress and strain tensors is independent of any choice of material constitutive law.

1.2.3 Elasticity

The constitutive equation of an elastic material is given in the form

$$\sigma = \mathbf{G}(\mathbf{F}), \tag{2.39}$$

where $\mathbf{G}$ is a *symmetric tensor-valued function*, defined on the space of deformation gradients $\mathbf{F}$. In general the form of $\mathbf{G}$ depends on the choice of reference configuration and $\mathbf{G}$ is referred to as the *response function* of the material *relative* to $\mathcal{B}_r$. For a given $\mathcal{B}_r$, therefore, the stress in $\mathcal{B}$ at a (material) point $\mathbf{X}$ depends only on the deformation gradient at $\mathbf{X}$ and not on the history of deformation. A material whose constitutive law has the form (2.39) is generally referred to as a *Cauchy elastic material.* Its specialization to the situation when there exists a strain-energy function will be considered in Section 1.2.4.

If the stress vanishes in $\mathcal{B}_r$ then

$$\mathbf{G}(\mathbf{I}) = \mathbf{O}, \tag{2.40}$$

and $\mathcal{B}_r$ is called a *natural configuration.* If the stress does not vanish in $\mathcal{B}_r$ then there is said to be *residual stress* in this configuration. In a residually-stressed configuration the traction must vanish at all points of the boundary, so that *a fortiori* residual stress is inhomogeneous in character. For detailed discussion of residual stress we refer to the work of Hoger and co-workers (see, for example, Hoger, 1985, 1986, 1993a, b and Johnson and Hoger 1993, 1995, 1998).

1.2.3.1 Objectivity

Suppose that a rigid-body deformation

$$\mathbf{x}^* = \mathbf{Q}\mathbf{x} + \mathbf{c} \tag{2.41}$$

is superimposed on the deformation $\mathbf{x} = \chi(\mathbf{X})$, where $\mathbf{Q}$ and $\mathbf{c}$ are constants, $\mathbf{Q}$ being a rotation tensor and $\mathbf{c}$ a translation vector. Then, the resulting deformation gradient, $\mathbf{F}^*$ say, is given by

$$\mathbf{F}^* = \mathbf{QF}. \tag{2.42}$$

For an elastic material with response function $\mathbf{G}$ relative to $\mathcal{B}_r$, the Cauchy stress tensor, $\boldsymbol{\sigma}^*$ say, associated with the deformation gradient $\mathbf{F}^*$ is $\boldsymbol{\sigma}^* = \mathbf{G}(\mathbf{F}^*)$.

Under the transformation (2.41) $\boldsymbol{\sigma}$ transforms according to the formula

$$\boldsymbol{\sigma}^* = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T. \tag{2.43}$$

The response function $\mathbf{G}$ must therefore satisfy the *invariance requirement*

$$\mathbf{G}(\mathbf{F}^*) \equiv \mathbf{G}(\mathbf{QF}) = \mathbf{QG}(\mathbf{F})\mathbf{Q}^T \tag{2.44}$$

for each deformation gradient $\mathbf{F}$ and *all* rotations $\mathbf{Q}$. This expresses the fact that the constitutive law (2.39) is *objective*. The terminology *material frame-indifference* is also used for this concept of objectivity (see, for example, Truesdell and Noll, 1965). In essence, this means that material properties are independent of superimposed rigid-body deformations.

A second-order Eulerian tensor, such as $\boldsymbol{\sigma}$, which satisfies the transformation rule (2.43) is said to be an (Eulerian) *objective second-order tensor*. We now expand on this notion slightly. Let $\phi, \mathbf{u}, \mathbf{T}$ be (Eulerian) scalar, vector and (second-order) tensor functions defined on $\mathcal{B}$. Let $\phi^*, \mathbf{u}^*, \mathbf{T}^*$ be the corresponding functions defined on $\mathcal{B}^*$, where $\mathcal{B}^*$ is obtained from $\mathcal{B}$ by the rigid deformation (2.41). The functions are said to be (Eulerian) *objective scalar, vector and tensor functions* (or fields) if, for all such deformations,

$$\phi^* = \phi, \quad \mathbf{u}^* = \mathbf{Qu}, \quad \mathbf{T}^* = \mathbf{QTQ}^T. \tag{2.45}$$

We observe that the density ρ is an example of an objective scalar function and that the normal vector $\mathbf{n}$, which appears in (2.8), and the traction vector $\mathbf{t}$, given by (2.26), are examples of objective vector functions, while the left Cauchy-green deformation tensor $\mathbf{B}$ is an objective tensor function.

It is important to distinguish between the behaviour of Lagrangian and Eulerian vector and tensor functions as far the definition of objectivity is concerned. The vector function $\mathbf{N}$, which is related to $\mathbf{n}$ by (2.8), and the right Cauchy-Green deformation tensor $\mathbf{C}$, given by (2.22), for example, are unchanged under the transformation (2.41). They are Lagrangian functions defined on $\mathcal{B}_r$. Thus, objectivity may equally well be defined in terms of Lagrangian functions. An objective Lagrangian (scalar, vector or tensor) function is one which is *unchanged* by the transformation (2.41). Other examples of objective Lagrangian tensors are the Biot and second Piola-Kirchhoff stress tensors defined in (2.36). Objective mixed tensors, such as $\mathbf{F}$, which are partly Lagrangian and partly Eulerian, change either as in (2.42) or its transpose. Thus, the nominal stress tensor $\mathbf{S}$, given by (2.28), transforms like $\mathbf{S}^* = \mathbf{S}\mathbf{Q}^T$ (for more detailed discussion, see Ogden, 1984b).

We mention here that Lagrangian vectors and tensors can be transformed into Eulerian vectors and tensors by appropriate 'push-forward' operations and this process is reversed by 'pull-back' transformations in the sense described in Marsden and Hughes (1994); see also Holzapfel (2000). The form of the push-forward and pull-back transformations depends on whether the vectors and tensors in question have covariant or contravariant character. For example, the push forward of the (covariant) Green strain tensor $\mathbf{E}$ is $\mathbf{F}^{-T}\mathbf{E}\mathbf{F}^{-1}$, which is an Eulerian strain tensor, while the push forward of the (contravariant) second Piola-Kirchhoff stress tensor $\mathbf{T}^{(2)}$ is $\mathbf{F}\mathbf{T}^{(2)}\mathbf{F}^T$, which is just J times the (Eulerian) Cauchy stress tensor. Partial push forward or pull back can be applied to either type of tensor to obtain mixed tensors or to mixed tensors to obtain Lagrangian or Eulerian tensors.

1.2.3.2 Material symmetry

Let $\boldsymbol{\sigma}$ be the stress in configuration $\mathcal{B}$, and let $\mathbf{F}$ and $\mathbf{F}'$ be the deformation gradients in $\mathcal{B}$ relative to two different reference configurations, $\mathcal{B}_r$ and $\mathcal{B}'_r$ respectively. We denote by $\mathbf{G}$ and $\mathbf{G}'$ the response functions relative to $\mathcal{B}_r$ and $\mathcal{B}'_r$, so that

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) = \mathbf{G}'(\mathbf{F}'). \tag{2.46}$$

Let $\mathbf{P} = \operatorname{Grad}\mathbf{X}'$ be the deformation gradient of $\mathcal{B}'_r$ relative to $\mathcal{B}_r$, where $\mathbf{X}'$ is the position vector of a point in $\mathcal{B}'_r$. Then

$$\mathbf{F} = \mathbf{F}'\mathbf{P}. \tag{2.47}$$

Substitution of (2.47) into (2.46) then gives $\mathbf{G}(\mathbf{F}'\mathbf{P}) = \mathbf{G}'(\mathbf{F}')$.

In general, the response of the material relative to $\mathcal{B}'_r$ differs from that relative to $\mathcal{B}_r$, i.e $\mathbf{G}' \neq \mathbf{G}$. However, for specific $\mathbf{P}$ we may have $\mathbf{G}' = \mathbf{G}$, in which case

$$\mathbf{G}(\mathbf{F}'\mathbf{P}) = \mathbf{G}(\mathbf{F}') \tag{2.48}$$

for all deformation gradients $\mathbf{F}'$ and for all such $\mathbf{P}$. Equation (2.46) then gives $\boldsymbol{\sigma} = \mathbf{G}(\mathbf{F}) = \mathbf{G}(\mathbf{F}')$, and, in order to calculate $\boldsymbol{\sigma}$, it is not necessary to distinguish between $\mathcal{B}_r$ and $\mathcal{B}'_r$.

The set of tensors $\mathbf{P}$ for which (2.48) holds forms a multiplicative group, called the *symmetry group of the material relative to* $\mathcal{B}_r$. This group characterizes the physical symmetry properties of the material.

Let $\mathbf{P}$ be the deformation gradient $\mathcal{B}_r \to \mathcal{B}'_r$, and now we do *not* assume that $\mathbf{P}$ is a member of the symmetry group. Then, if $\mathcal{G}$ is the symmetry group of the material relative to $\mathcal{B}_r$ and $\mathcal{G}'$ that relative to $\mathcal{B}'_r$ these groups are related according to *Noll's rule*

$$\mathcal{G}' = \mathbf{P}\mathcal{G}\mathbf{P}^{-1}. \tag{2.49}$$

Clearly, for the special case in which $\mathbf{P} \in \mathcal{G}$, we have $\mathcal{G}' = \mathcal{G}$.

1.2.3.3 Isotropic elasticity

If $\mathcal{G}$ is the proper orthogonal group then the material is said to be *isotropic relative to* $\mathcal{B}_r$, and then

$$\sigma = \mathbf{G}(\mathbf{FQ}) = \mathbf{G}(\mathbf{F}) \tag{2.50}$$

for all proper orthogonal $\mathbf{Q}$ and for every deformation gradient $\mathbf{F}$. Physically, this means that the response of a small specimen of material is independent of its orientation in $\mathcal{B}_r$.

Before proceeding further we require some definitions and results relating to isotropic functions of a second-order tensor. Firstly, the scalar function $\phi(\mathbf{T})$ of a *symmetric* second-order tensor $\mathbf{T}$ is said to be an *isotropic function* of $\mathbf{T}$ if

$$\phi(\mathbf{QTQ}^T) = \phi(\mathbf{T}) \tag{2.51}$$

for all orthogonal tensors $\mathbf{Q}$. An isotropic scalar-valued function of $\mathbf{T}$ is also called a *scalar invariant* of $\mathbf{T}$. It may easily be checked that the *principal invariants* of $\mathbf{T}$, defined by

$$I_1(\mathbf{T}) = \mathrm{tr}\,(\mathbf{T}), \quad I_2(\mathbf{T}) = \frac{1}{2}[I_1(\mathbf{T})^2 - \mathrm{tr}\,(\mathbf{T}^2)], \quad I_3(\mathbf{T}) = \det \mathbf{T}, \tag{2.52}$$

are scalar invariants in accordance with the definition (2.51). It may be shown that $\phi(\mathbf{T})$ is a scalar invariant of $\mathbf{T}$ if and only if it is expressible as a function of $I_1(\mathbf{T}), I_2(\mathbf{T}), I_3(\mathbf{T})$.

Secondly, suppose that $\mathbf{G}(\mathbf{T})$ is a symmetric second-order tensor function of $\mathbf{T}$. Then, $\mathbf{G}(\mathbf{T})$ is said to be an *isotropic tensor function* of $\mathbf{T}$ if

$$\mathbf{G}(\mathbf{QTQ}^T) = \mathbf{QG}(\mathbf{T})\mathbf{Q}^T \tag{2.53}$$

for all orthogonal $\mathbf{Q}$. Consequences of this are (i) if $\mathbf{G}(\mathbf{T})$ is isotropic then its eigenvalues are scalar invariants of $\mathbf{T}$, (ii) $\mathbf{G}(\mathbf{T})$ is coaxial with $\mathbf{T}$, i.e.

$$\mathbf{G}(\mathbf{T})\mathbf{T} = \mathbf{TG}(\mathbf{T}), \tag{2.54}$$

and (iii) $\mathbf{G}(\mathbf{T})$ is isotropic if and only if it has the representation

$$\mathbf{G}(\mathbf{T}) = \phi_0\mathbf{I} + \phi_1\mathbf{T} + \phi_2\mathbf{T}^2, \tag{2.55}$$

where ϕ_0, ϕ_1, ϕ_2 are scalar invariants of $\mathbf{T}$ and hence functions of $I_1(\mathbf{T})$, $I_2(\mathbf{T})$, $I_3(\mathbf{T})$.

The choice $\mathbf{Q} = \mathbf{R}^T$ and use of the polar decomposition $\mathbf{F} = \mathbf{VR}$ in (2.50) gives

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V}). \tag{2.56}$$

We then obtain

$$\mathbf{QG}(\mathbf{V})\mathbf{Q}^T = \mathbf{G}(\mathbf{QVQ}^T) \tag{2.57}$$

for all proper orthogonal $\mathbf{Q}$. In fact, since $\mathbf{Q}$ occurs twice on each side of (2.57), allowing $\mathbf{Q}$ to be improper orthogonal does not affect (2.57), which then states that $\mathbf{G}(\mathbf{V})$ is an isotropic function of $\mathbf{V}$ in accordance with the definition (2.53).

In particular, for an *isotropic elastic material*, $\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V})$ *is coaxial with* $\mathbf{V}$, i.e. with the Eulerian principal axes, and we therefore have

$$\boldsymbol{\sigma} = \mathbf{G}(\mathbf{V}) = \phi_0\mathbf{I} + \phi_1\mathbf{V} + \phi_2\mathbf{V}^2, \tag{2.58}$$

where ϕ_0, ϕ_1, ϕ_2 are scalar invariants of $\mathbf{V}$, i.e. functions of

$$i_1 = I_1(\mathbf{V}) = \mathrm{tr}\,(\mathbf{V}) = \lambda_1 + \lambda_2 + \lambda_3, \tag{2.59}$$

$$i_2 = I_2(\mathbf{V}) = \frac{1}{2}[i_1^2 - \mathrm{tr}\,(\mathbf{V}^2)] = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \tag{2.60}$$

$$i_3 = I_3(\mathbf{V}) = \det \mathbf{V} = \lambda_1\lambda_2\lambda_3, \tag{2.61}$$

where the expressions have also been given in terms of the principal stretches and the notation i_1, i_2, i_3 has been introduced specifically for the principal invariants of $\mathbf{V}$ (and hence of $\mathbf{U}$). Alternatively, we may write

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \sigma_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.62}$$

where

$$\sigma_i = \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2 \qquad i \in \{1, 2, 3\}, \tag{2.63}$$

and this allows us to introduce the *scalar response function* g, such that

$$\sigma_i = g(\lambda_i, \lambda_j, \lambda_k) = g(\lambda_i, \lambda_k, \lambda_j) \equiv \phi_0 + \phi_1\lambda_i + \phi_2\lambda_i^2, \tag{2.64}$$

where (i, j, k) is permutation of $(1, 2, 3)$.

The expansion (2.58) may be written, equivalently, in terms of $\mathbf{B} = \mathbf{V}^2$. For example,

$$\boldsymbol{\sigma} = \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{B}^2, \tag{2.65}$$

or

$$\boldsymbol{\sigma} = \beta_0\mathbf{I} + \beta_1\mathbf{B} + \beta_{-1}\mathbf{B}^{-1}, \tag{2.66}$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_{-1}$ are scalar invariants of $\mathbf{B}$ (and hence of $\mathbf{V}$); see, for example, Beatty (1987). Connections between these different coefficients are determined by using the Cayley-Hamilton theorem in the form

$$\mathbf{V}^3 - i_1\mathbf{V}^2 + i_2\mathbf{V} - i_3\mathbf{I} = \mathbf{O} \tag{2.67}$$

or its counterpart for $\mathbf{B}$. It is convenient in what follows to use the standard notation I_1, I_2, I_3 for the principal invariants of $\mathbf{B}$ (also of $\mathbf{C}$). Thus, specifically, we write

$$I_1 = I_1(\mathbf{B}) = \mathrm{tr}\,(\mathbf{B}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{2.68}$$

$$I_2 = I_2(\mathbf{B}) = \frac{1}{2}[I_1^2 - \mathrm{tr}\,(\mathbf{B}^2)] = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \tag{2.69}$$

$$I_3 = I_3(\mathbf{B}) = \det\mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.70}$$

In view of the connection (2.28) between $\mathbf{S}$ and $\boldsymbol{\sigma}$ we may also define the response function, $\mathbf{H}$ say, associated with $\mathbf{S}$ (relative to $\mathcal{B}_r$) by

$$\mathbf{S} = \mathbf{H}(\mathbf{F}) \equiv J\mathbf{F}^{-1}\mathbf{G}(\mathbf{F}). \tag{2.71}$$

The objectivity requirement (2.44) then becomes

$$\mathbf{H}(\mathbf{QF}) = \mathbf{H}(\mathbf{F})\mathbf{Q}^T. \tag{2.72}$$

A corresponding change for the material symmetry transformation (2.48) can be written down, and, in particular, for an isotropic elastic solid, we have

$$\mathbf{H}(\mathbf{FQ}) = \mathbf{Q}^T\mathbf{H}(\mathbf{F}). \tag{2.73}$$

Moreover, it follows from (2.73) that

$$\mathbf{H}(\mathbf{F}) = \mathbf{H}(\mathbf{U})\mathbf{R}^T = \mathbf{R}^T\mathbf{H}(\mathbf{V}), \tag{2.74}$$

with $\mathbf{H}(\mathbf{U})$ being symmetric and coaxial with $\mathbf{U}$.

1.2.3.4 Internal constraints

In Section 1.2.1 the (internal) constraints of incompressibility and inextensibility were mentioned. More generally, a single constraint may be written in the form

$$C(\mathbf{F}) = 0, \tag{2.75}$$

where C is a scalar function. Equation (2.75) holds for all possible deformation gradients $\mathbf{F}$. For the incompressibility and inextensibility constraints we have, respectively,

$$C(\mathbf{F}) = \det\mathbf{F} - 1, \qquad C(\mathbf{F}) = \mathbf{M}\cdot(\mathbf{F}^T\mathbf{F}\mathbf{M}) - 1. \tag{2.76}$$