Domain Conditions in Social Choice Theory

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CHAPTER 1

INTRODUCTION

The theory of social choice is abundant with impossibility theorems. The simplest impossibility result is probably the 'paradox of voting' which has been known for a long time. Imagine that there is a society comprising just three individuals who have to decide on whether to adopt policy x or policy y or policy z in order to increase economic welfare in their small community. Anticipating that their individual preferences will by no means be unanimous, the three members of society have agreed to use the simple majority decision rule as their method of aggregation. Let individual 1 prefer x to y, y to z and x to z, individual 2 prefer y to z, z to x and y to x, and individual 3 prefer z to x, x to y and z to y. Having applied the simple majority rule, the three persons obtain the following result: x is socially preferred to y, y to z and z to x. Obviously, each policy is dominated by one of the other two policies by a majority of two to one. What should be done in this situation? This is a difficult question indeed. The proposal to determine a sequence of pairwise decisions is no way out of the impasse, for the three individuals can be expected to disagree sharply on which pair of alternatives should be the first in the sequence of pairwise choices. Actually, in the latter part of the proof of his well-known impossibility result, Arrow (1951, 1963) used 'an appropriate adaptation of the paradox of voting' (1963, p. 100). Arrow's negative result is, of course, much more general than the cyclical structure of majority preference depicted above but with some justification perhaps, the paradox of voting can be described as the tip of an iceberg, the iceberg standing for a quite general axiomatic structure and its unviability.[1]

[1] As Sen (1985, p. 1769) writes, 'we must reject seeing the "Arrow problem" merely as a generalization of the paradox of voting. It is

According to McLean and London (1990), the roots of the theory of collective choice can be traced back to the end of the thirteenth century (possibly earlier than that[2]) when Ramon Lull designed two voting procedures that have a striking resemblance to what has 500 years later become known as the 'Borda method' and the 'Condorcet principle'. In his novel *Blanquerna* (around 1283) Lull made Natana explain a new electoral method to all the sisters in her nunnery, a method consisting of exhaustive pairwise comparisons, i.e., each candidate is compared to every other candidate under consideration. However, Natana (or Lull) does not advocate the choice of the 'Condorcet winner'[3] but the choice of the candidate who receives the highest number of votes in the aggregate of the pairwise comparisons. This procedure is identical to a method proposed by Borda in 1770 which, as was shown by Borda (1781) himself, must yield the same result as his well-known rank-order method.

The second procedure, devised in 1299, was put forward by Lull in his treatise *De Arte Eleccionis*. Here a successive voting rule is proposed that ends up with a 'Condorcet winner', if there exists one. Since not every logically possible pairwise comparison is made in determining the winner, the suggested procedure does not necessarily detect the existence of cycles. In the case that a cycle occurs, the outcome depends directly on the selected path of pairwise comparisons, but it is not clear whether Lull was aware of this fact.

much more than that.' On this and other paradoxes in economics, see De Marchi (1987).

[2] McLean and London refer to a letter by Pliny the Younger (around 90AD) in which secret ballots in the Roman Senate are discussed (see also Radice 1969, I, pp. 230–5). In that letter voting among three or more candidates is not mentioned. However, in a letter to Titius Aristo, voting over three distinct alternatives is discussed. Plinius describes a situation where one group of persons changes its preferences by dropping its preferred option, thereby generating an outcome that would not have been reached under pairwise majority decision over the original set of options (I owe this reference to Salvador Barberà).

[3] For chronological reasons, we have decided to put the concepts of Condorcet winner, Condorcet principle and Borda method in inverted commas here.

Nicolaus Cusanus had read *De Arte Eleccionis* (see Honecker 1937) but he rejected Lull's 'Condorcet procedure' and proposed instead a 'Borda rank-order method' with secret voting.[4] McLean and London indicate that Cusanus rejected Lull's 'Condorcet principle' for deeper reasons and not out of misunderstanding. In 1688 Pufendorf published his work *De jure naturae et gentium*, in which a few pages were devoted to various decision schemes such as majority and plurality rules.[5] Pufendorf, as well as Lull and Cusanus before him, explicitly mention the issue of telling the truth in an election.[6] The possibility of manipulation within collective choice processes is a phenomenon which has been receiving a lot of attention since the mid-1970s after the important findings of Gibbard (1973), Pattanaik (1973) and Satterthwaite (1973, 1975).

Much better known than Lull's, Cusanus' and Pufendorf's writings are the works by de Borda (1781) and the Marquis de Condorcet (1785). Condorcet extensively discussed the election of candidates under the majority rule. He was probably the first to demonstrate the existence of cyclical majorities for particular preference profiles (but nowhere did he discuss the symmetrical structure of our introductory example).[7] Condorcet called these situations 'contradictory', for in the case of three alternatives, let's say, any two of the propositions lead to

[4] See McLean and London (1990) for further references. Cusanus (1434) dealt with the election of a Holy Roman Emperor.

[5] For more details on Pufendorf's work see Lagerspetz (1986).

[6] Cusanus said in his *De concordantia catholica* (around 1434) that 'elections could be said to be disgracefully rigged by unjust pacts' (see McLean and London 1990).

[7] Though we have to concede that he was close to our 3×3 formulation, interestingly enough in the context of voting on economic policy: whether any restriction placed on commerce is an injustice or whether restrictions placed through general laws or by particular orders can be just (I am grateful to Emma Rothschild for this observation). According to Baker (1975), the paradoxical result of a voting cycle as in our introductory example was first properly called the 'Condorcet effect' by Guilbaud (1952).

a proposition which contradicts the third.[8] Condorcet proposed a resolution scheme for the case of cyclical majorities.[9] His arguments remained fragmentary, however, for the situation of more than three candidates.[10] Almost one hundred years later, Dodgson (1876) explicitly dealt with the case of cyclical majorities under various voting schemes but came to the conclusion that if there are persistent majority cycles there ought to be 'no Election' if this is an allowable outcome.[11]

To the best of our knowledge, none of the authors mentioned above wrote about a solution to the problem of cyclical majorities via restricting 'the shape'of individual preference orderings and 'the composition' of preference profiles.[12] Domain conditions of various forms for various collective choice rules are the topic of this monograph. In

[8] Black (1958, p. 167) finds Condorcet's use of the word 'contradictory' unfortunate. We fully agree with him when he writes that 'the danger is that describing these results as ''contradictory'' ... might suggest that the group has a scale of valuations which is the same in kind as that of the individual, which would be false: the individual values, the group does not; it reaches decisions through some procedure in voting'. On this point, see also De Marchi (1987) who claims that economists tend to employ 'micro-motives to account for aggregate relations whose entities they cannot explain'.

[9] Condorcet argued that in a case where, for example, a majority prefers x to y, y to z and z to x (as in our introductory example), the proposition with the smallest majority should be deleted and that alternative should be chosen which comes out as the winner under the two remaining propositions. Unfortunately, under the circumstances of our example above, Condorcet's suggestion would not help us to decide which of the three propositions we should delete.

[10] Part II of Black's book (1958) is an excellent source for further information on the history of the mathematical theory of collective decisions. See also Riker's (1986) historical remarks on weighted voting games.

[11] See Black (1958), pp. 224–34.

[12] Can one take the following quotations from Dodgson (see Black 1958, p. 225) as an indication that the author had a restriction on individual preferences in mind? 'When the issues to be further debated consist of, or have been reduced to, a single cycle, the Chairman shall inform the meeting how many alterations of votes each issue requires to give it a majority over every other separately ... If, when the majorities are found to be cyclical, any elector wishes to alter his paper, he may do so.'

his seminal work Arrow required that the range of the collective choice rule he was analysing (the social welfare function) be restricted to the set of orderings over the set of alternatives. One may find a lower degree of collective rationality quite acceptable. One can think of extending the range of the collective choice rule to include those social preference relations that are not orderings, but which always generate a nonempty choice set (the concept of a social decision function). Furthermore, Arrow and many other writers required the individual weak preference relations to be orderings, but one can think of arguments for weakening this condition and demand that the individual preference relations be reflexive, connected and quasi-transitive, not fully transitive. Both suggestions will be examined in detail. Furthermore we shall consider a variety of aggregation mechanisms, not only the simple majority decision rule but also – among others – special majority rules, multi-stage majority rules, simple games of various forms, social welfare functions and stable group decision functions.

The literature on domain conditions for collective choice rules can be split up into two large classes. The contributions to the first class study the aggregation problem for arbitrary finite sets of discrete alternatives. These options can be political parties or candidates representing these parties; these alternatives can also stand for particular economic and (or) social programmes. The contributions to the second class assume that the set of options has a topological structure. Most authors in this category suppose that the choice space is the *n*-dimensional Euclidean space. Within the first class, a further distinction can be made: The domain conditions with respect to the individual preference relations either have the characteristic of being exclusion conditions (particular individual preference relations are not permitted to be held by any member of society or particular preference relations are not allowed to occur in the presence of other preference relations), or the domain conditions admit all logically possible individual preference orderings, for example, but make certain requirements as to the distribution of individuals over these orderings.

The best-known example for the first type of restrictions is Black's (1948) condition of single-peaked preferences[13] which is depicted below in figure 1.1 for three alternatives *x*, *y* and *z* and three individuals. The vertical axis just indicates the order of preference – no cardinality is involved. We have chosen a particular ordering of the alternatives along the horizontal axis. However, there is flexibility with respect to the choice of this ordering. The lines between the symbols have no meaning. They simply help to interpret the structure of points as single-peaked.

As one can see from figure 1.1, a single-peaked graph is one which changes its direction at most once, when running from up to down. At an interpretative level, preferences are single-peaked if 'more' is strictly preferred to 'less' up to a point, and 'less' to 'more' beyond that point. Given a set of single-peaked curves, this restriction on preferences is rather easy to interpret. Let the committee's decision be with regard to the price of a new product. Each member of the committee will, in order to shape his (her) opinion on this matter, initially try to find

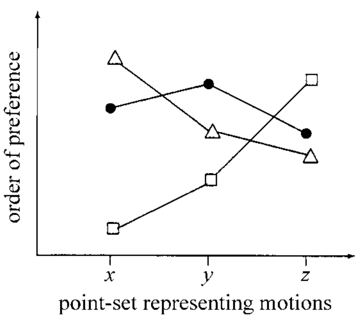

Figure 1.1.

[13] As a matter of fact, the idea of single-peakedness was developed independently by Arrow (though in print, Black was first). On this point, see the interview with K. J. Arrow by Kelly (1987).

out which is the optimal price for him (her). Once the optimal price has been fixed, 'the further any proposal departs from it on the one side or the other, the less he [she] will favour it' (Black, 1958, p. 9). Arrow (1951, 1963, p. 76) gives another example where Black's assumption seems to have been satisfied: the party structure of prewar European parliaments with a clear Left–Right ordering of the parties.

Let us take a second look at figure 1.1. One can easily see that alternative *y* lies between *x* and *z* in two of the three orderings and is 'best' in one ordering. Therefore, *y* is nowhere considered as 'the worst' of the three given options. This different, though an analytically equivalent aspect of single-peakedness, due to Sen (1966), uses a perspective which is quite different from the view on which Black and Arrow focused. This reinterpretation opened the door for various generalizations of the original condition which we shall discuss later on. Using Black's domain condition Arrow has shown that the simple majority decision rule generates a transitive social preference relation for any number of alternatives provided that the number of individuals is odd and the property of single-peakedness is fulfilled for every triple of alternatives.

The distributional requirements have been studied mainly for the method of majority decision. It is worth mentioning already at this point that there exists a logical relationship between some of the restrictions in this set of requirements and some of the exclusion conditions. Several of the authors in the area of distributional domain conditions have argued that the problem of the existence of a transitive social preference relation under the majority rule should be best studied for artificially constructed societies with so-called reduced preference profiles. The idea is as follows: Under the method of majority decision the two orderings '*x* preferred to *y* and *y* preferred to *z*' and '*z* preferred to *y* and *y* preferred to *x*' ($xPyPz$ and $zPyPx$, for short) cancel out as they are inverse to each other. Imagine that we have a society of six individuals with the following orderings: xP_1yP_1z, yP_2zP_2x, zP_3xP_3y, yP_4xP_4z, xP_5zP_5y, and zP_6yP_6x. Applying the majority rule we obtain the result that the society is indifferent among the three options. If we now add a seventh individual who has one of the six orderings above, this individual's ordering will determine the collective ordering. No majority cycle arises though, quite obviously, all the exclusion conditions are violated.

While the larger part of this monograph examines finite sets of discrete alternatives, the final chapter considers the aggregation issue for choice spaces with a topological structure. In various economic problems, the possible choices constitute a set of points in some appropriately defined multi-dimensional continuous choice space. For decisions on the composition of the federal budget, for example, or decisions on the production of various public goods the n-dimensional Euclidean space may be the appropriate space to consider. Another example would be choices over a set of social states that are characterized by numerical values only such as the unemployment rate, the inflation rate and the federal deficit, let's say. Individual preferences are represented by quasi-concave, differentiable utility functions defined over this space. Note that in those cases where the individual utility functions are defined on a compact interval of the real line, or where, for example, the north-east boundary of a feasible set is the focus of attention, as in many two-dimensional constrained maximization problems, quasi-concavity of the utility functions can be viewed as a generalization of Black's single-peakedness property. We shall discuss what impact the transition from finite sets of discrete alternatives to multi-dimensional topological spaces has on the issues of domain restriction and the existence of social choice rules. We also want to ask whether a logical relationship exists between the former, the standard approach so to speak, and the latter more recent method.

We should mention that Arrow chose a nontopological framework for his analysis and many papers that were written in response to his impossibility result did exactly the same. Continuity was not considered to be a relevant property. Of course, Arrow's impossibility result which we shall discuss briefly in chapter 2 can be formulated in n-dimensional continuous space and we would like to draw the reader's attention particularly to parts of chapter 4.1 and chapter 6.1 below (the reader may also wish to consult Inada 1964a for the existence resp. nonexistence of welfare functions in n-dimensional space). Note, for example, that spatial voting models, which we shall not discuss in this monograph, are often defined in two-dimensional Euclidean space. Both simple majority voting as an aggregation scheme and single-peakedness as a possible domain restriction are concepts that are directly applicable in such a space.

Is there any justification to consider domains that are restricted in the sense of not allowing particular individual preference relations to occur? We already mentioned that in the political arena a clear Left–Right ordering of the parties appears reasonable. If some individual is leaning toward the political Right, he or she will most probably prefer a candidate of this group to a candidate from the centre party and the latter most probably to a candidate from the left wing. Single-peakedness also makes good sense in various location problems when, for example, people want to live as close as possible to the city centre or students want to be as close as possible to the university campus.

In other instances, individuals wish to be as far as possible from a refuse disposal site or a coal-fired power station. Coming back to the Left–Right ordering of political parties, it is sometimes argued that people who tend to be extremists either vote for the extreme Right or the extreme Left, while they dislike parties in the middle of the spectrum. These cases are covered by so-called single-caved preferences, the mirror image of single-peakedness.

A common historical background or a common class background may bring about a fair amount of similarity among the individuals' preferences, and if a particular society consists mainly of two classes, the possibility that some preference relations will nowhere be encountered is relatively high. However, for a group of n individuals with $n-1$ persons having single-peaked preferences and only one individual showing nonsingle-peaked preferences, cyclical majorities may arise. Therefore, a great deal of caution is advisable. On the other hand, Sen (1970, p. 165) is certainly right when he says that 'individual preferences are determined not by turning a roulette wheel over all possible alternatives'. Specific economic forces, education, but also, admittedly, manipulation are among those factors which can significantly shape individual preference relations. Therefore, in our opinion, it makes good sense to investigate the aggregation problem when only some, but not all logically possible individual preference relations occur.[14]

[14] A different perspective which, however, will not be followed in this monograph, was brought forward in an empirical investigation by Feld and Grofman (1987). They argued that requiring *each and every individual* in society to follow single-peakedness is demanding too much. What is sufficient for a transitive majority preference relation of the society as a whole is that *each subgroup* within society satisfy what they call the condition of ideologically ordered margins. The authors were investigating how individuals ranked the four Presidential candidates Carter, Kennedy, Ford and Reagan.