Maths

A Student's Survival Guide

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Contents

I have split the chapters up in the following way so that you can easily find particular topics. Also, it makes it easy for me to tell you where to go if you need help, and easy for you to find this help.

Introduction

I have written this book mainly for students who will need to apply maths in science or engineering courses. It is particularly designed to help the foundation or first year of such a course to run smoothly but it could also be useful to specialist maths students whose particular choice of A-level or pre-university course has meant that there are some gaps in the knowledge required as a basis for their University course. Because it starts by laying the basic groundwork of algebra it will also provide a bridge for students who have not studied maths for some time.

The book is written in such a way that students can use it to sort out any individual difficulties for themselves without needing help from their lecturers.

A message to students

I have made this book as much as possible as though I were talking directly to you about the topics which are in it, sorting out possible difficulties and encouraging your thoughts in return. I want to build up your knowledge and your courage at the same time so that you are able to go forward with confidence in your own ability to handle the techniques which you will need. For this reason, I don't just tell you things, but ask you questions as we go along to give you a chance to think for yourself how the next stage should go. These questions are followed by a heavy rule like the one below.

It is very important that you should try to answer these questions yourself, so the rule is there to warn you not to read on too quickly.

I have also given you many worked examples of how each new piece of mathematical information is actually used. In particular, I have included some of the off-beat non-standard examples which I know that students often find difficult.

To make the book work for you, it is vital that you do the questions in the exercises as they come because this is how you will learn and absorb the principles so that they become part of your own thinking. As you become more confident and at ease with the methods, you will find that you enjoy doing the questions, and seeing how the maths slots together to solve more complicated problems.

Always be prepared to think about a problem and have a go at it – don't be afraid of getting it wrong. Students very often underrate what they do themselves, and what they *can* do. If something doesn't work out, they tend to think that their effort was of no worth *but this is not true*. Thinking about questions for yourself is how you learn and understand what you are doing. It is much better than just following a template which will only work for very similar problems and *then* only if you recognise them. If you really understand what you are doing you will be able to apply these ideas in later work, and this is important for you.

Because you may be working from this book on your own, I have given detailed solutions to most of the questions in the exercises so that you can sort out for yourself any problems that you may have had in doing them. (Don't let yourself be tempted just to read through my solutions – you will do infinitely better if you write your own solutions first. This is the most important single piece of advice which I can give you.) Also, if you are stuck and have to look at my solution, don't just read through the whole of it. Stop reading at the point that gets you unstuck and see if you can finish the problem yourself.

I have also included what I have called thinking points. These are usually more open-ended questions designed to lead you forward towards future work.

If possible, talk about problems with other students; you will often find that you can help each other and that you spark each other's ideas. It is also very sensible to scribble down your thoughts as you go along, and to use your own colour to highlight important results or particular parts of drawings. Doing this makes you think about which are the important bits, and gives you a short-cut when you are revising.

There are some pitfalls which many students regularly fall into. These are marked

to warn you to take particular notice of the advice there. You will probably recognise some old enemies!

It often happens in maths that in order to understand a new topic you must be able to use earlier work. I have made sure that these foundation topics are included in the book, and I give references back to them so that you can go there first if you need to. I have linked topics together so that you can see how one affects another and how they are different windows onto the same world. The various approaches, visual, geometrical, using the equations of algebra or the arguments of calculus, all lead to an understanding of how the fundamental ideas interlock. I also show you wherever possible how the mathematical ideas can be used to describe the physical world, because I find that many students particularly like to know this, and indeed it is the main reason why they are learning the maths. (Much of the maths is very nice in itself, however, and I have tried to show you this.)

I have included in some of the thinking points ideas for simple programs which you could write to investigate what is happening there. To do this, you would need to know a programming language and have access to either a computer or programmable calculator. I have also suggested ways in which you can use a graph-sketching calculator as a fast check of what happens when you build up graphs from combinations of simple functions. Although these suggestions are included because I think you would learn from them and enjoy doing them, it is not necessary to have this equipment to use this book.

Much of the book has grown from the various comments and questions of all the students I have taught. It is harder to keep this kind of two-way involvement with a printed book but no longer impossible thanks to the Web. I would be very interested in your comments and questions and grateful for your help in spotting any mistakes which may have slipped through my checking. You can contact me by using my email address of jenolive@netcomuk.co.uk and I look forward to putting little additions on the Web, sparked by your thoughts. My website is at http://www.netcomuk.co.uk/~jenolive

Finally, I hope that you will find that this book will smooth your way forward and help you to enjoy all your courses.

Jenny Olive

1 Basic algebra: some reminders of how it works

In many areas of science and engineering, information can be made clearer and more helpful if it is thought of in a mathematical way. Because this is so, algebra is extremely important since it gives you a powerful and concise way of handling information to solve problems. This means that you need to be confident and comfortable with the various techniques for handling expressions and equations.

The chapter is divided up into the following sections.

1.A Handling unknown quantities

1.A.(a) Where do you start? Self-test 1

All the maths in this book which is directly concerned with your courses depends on a foundation of basic algebra. In case you need some extra help with this, I have included two revision sections at the beginning of this first chapter. Each of these sections starts with a short self-test so that you can find out if you need to work through it.

It's important to try these if you are in any doubt about your algebra. You have to build on a firm base if you are to proceed happily; otherwise it is like climbing a ladder which has some rungs missing, or, more dangerously, rungs which appear to be in place until you tread on them.

Self-test 1

Answer each of the following short questions.

(A) Find the value of each of the following expressions if $a = 3$, $b = 1$, $c = 0$ and $d = 2$.
(1) a^2 (2) b^2 (3) $ab + d$ (4) $a(b + d)$ (5) $2c + 3d$
(6) $2a^2$ (7) $(2a)^2$ (8) $4ab + 3bd$ (9) $a + bc$ (10) d^3

(B) Find the values of each of the following expressions if $x = 2$, $y = -3$, $u = 1$, $v = -2$, $w = 4$ and $z = -1$.
(1) $3xy$ (2) $5vy$ (3) $2x + 3y + 2v$ (4) v^2 (5) $3z^2$
(6) $w + vy$ (7) $2x - 5vw$ (8) $2y - 3v + 2z - w$ (9) $2y^2$ (10) z^3

(C) Simplify (that is, write in the shortest possible form).
(1) $3p - 2q + p + q$ (2) $3p^2 + 2pq - q^2 - 7pq$ (3) $5p - 7q - 2p - 3q + 3pq$

(D) Multiply out the following expressions.
(1) $5(2g + 3h)$ (2) $g(3g - 2h)$ (3) $3k^2(2k - 5m + 2n)$ (4) $3k - (2m + 3n - 5k)$

(E) Factorise the following expressions.
(1) $3x^2 + 2xy$ (2) $3pq + 6q^2$ (3) $5x^2y - 7xy^2$

Here are the answers. (Give yourself one point for each correct answer, which gives a maximum possible score of 30.)

(A) (1) 9 (2) 1 (3) 5 (4) 9 (5) 6 (6) 18 (7) 36 (8) 18 (9) 3 (10) 8

(B) (1) –18 (2) 30 (3) –9 (4) 4 (5) 3 (6) 10 (7) 44 (8) –6 (9) 18 (10) –1

(C) (1) $4p - q$ (2) $3p^2 - 5pq - q^2$ (3) $3p - 10q + 3pq$

(D) (1) $10g + 15h$ (2) $3g^2 - 2gh$ (3) $6k^3 - 15k^2m + 6k^2n$ (4) $8k - 2m - 3n$

(E) (1) $x(3x + 2y)$ (2) $3q(p + 2q)$ (3) $xy(5x - 7y)$

If you scored anything less than 25 points then I would advise you to work through Section 1.A. If you made just the odd mistake, and realised what it was when you saw the answer, then go ahead to Section 1.B. If you are in any doubt, it is best to go through Section 1.A. now; these are your tools and you need to feel happy with them.

1.A.(b) A mind-reading explained

Much of what was tested above can be shown in the handling of the following. Try it for yourself. (You may have met this apparently mysterious kind of mind-reading before.)

(1) Think of a number between 1 and 10. (A small number is easier to use.)
(2) Add 3 to it.
(3) Double the number you have now.
(4) Add the number you first thought of.
(5) Divide the number you have now by 3.
(6) Take away the number you first thought of.
(7) The number you are thinking of now is . . . 2!

How can we lay bare the bones of what is happening here, so that we can see how it is possible for me to know your final answer even though I don't know what number you were thinking of at the start?

It is easier for me to keep track of what is happening, and so be able to arrange for it to go the way I want, if I label this number with a letter. So suppose I call it x. Suppose also that your number was 7 and we can then keep a parallel track of what goes on.

	You	**Me**
(1)	7	x
(2)	10	$x + 3$ (My unknown number plus 3.)
(3)	20	$2(x + 3) = 2x + 6$ (Each of these show the doubling.)
(4)	27	$2x + 6 + x = 3x + 6$ (I add in the unknown number.)
(5)	9	$\frac{3x + 6}{3} = x + 2$ (The whole of $3x + 6$ is divided by 3.)
(6)	2	2 (The x has been taken away.)

Both your 7 and my x have been got rid of as a result of this list of instructions.

My list uses algebra to make the handling of an unknown quantity easier by tagging it with a letter. It also shows some of the ways in which this handling is done.

1.A.(c) Some basic rules

There are certain rules which need to be followed in handling letters which are standing for numbers. Here I remind you of these.

Adding

$a + b$ means quantity a added to quantity b.

$a + a + b + b + b = 2a + 3b$. Here, we have twice the first quantity and three times the second quantity added together. There is no shorter way of writing $2a + 3b$ unless we know what the letters are standing for.

We could equally have said $b + a$ for $a + b$, and $3b + 2a$ for $2a + 3b$. It doesn't matter what order we do the adding in.

Multiplying

ab means $a \times b$ (that is, the two quantities multiplied together) and the letters are usually, but not always, written in alphabetical order.

In particular, $a \times 1 = a$, and $a \times 0 = 0$.
$5ab$ would mean $5 \times a \times b$.

It doesn't matter what order we do the multiplying in, for example $3 \times 5 = 5 \times 3$.

Working out powers

If numbers are multiplied by themselves, we use a special shorthand to show that this is happening.

a^2 means $a \times a$ and is called a squared.
a^3 means $a \times a \times a$ and is called a cubed.
a^n means a multiplied by itself with n lots of a and is called a to the power n.

Little raised numbers, like the 2, 3 and n above, are called **powers** or **indices**. Using these little numbers makes it much easier to keep a track of what is happening when we multiply. (It was a major breakthrough when they were first used.) You can see why this is in the following example.

Suppose we have $a^2 \times a^3$.
Then $a^2 = a \times a$ and $a^3 = a \times a \times a$ so $a^2 \times a^3 = a \times a \times a \times a \times a = a^5$.
The powers are added. (For example, $2^2 \times 2^3 = 4 \times 8 = 32 = 2^5$.)

> We can write this as a general rule.
>
> $$a^n \times a^m = a^{n+m}$$
>
> where a stands for any number except 0 and n and m can stand for any numbers.

In this section, n and m will only be standing for positive whole numbers, so we can see that they would work in the same way as the example above.

To make the rule work, we need to think of a as being the same as a^1. Then, for example, $a \times a^2 = a^1 \times a^2 = a^3$ which fits with what we know is true, for example $2 \times 2^2 = 2^3$ or $2 \times 4 = 8$.

Also, this rule for adding the powers when multiplying only works if we have powers of the same number, so $2^2 \times 2^3 = 2^5$ and $7^2 \times 7^3 = 7^5$ but $2^2 \times 7^3$ cannot be combined as a single power.

If we have numbers and different letters, we just deal with each bit separately, so for example $3a^2b \times 2ab^3 = 6a^3b^4$.

Working out mixtures – using brackets

$a + bc$ means quantity a added to the result of multiplying b and c. The multiplication of b and c must be done before a is added.

If $a = 2$ and $b = 3$ and $c = 4$ then $a + bc = 2 + 3 \times 4 = 2 + 12 = 14$.

If we want a and b to be added first, and the result to be multiplied by c, we use a bracket and write $(a + b)c$ or $c(a + b)$, as the order of the multiplication does not matter. This gives a result of $5 \times 4 = 4 \times 5 = 20$.

A bracket collects together a whole lot of terms so that the same thing can be done to all of them, like corralling a lot of sheep, and then dipping them. So $a(b + c)$ means $ab + ac$. The a multiplies every separate item in the bracket.

Similarly, $2x(x + y + 3xy) = 2x^2 + 2xy + 6x^2y$. The brackets show that everything inside them is to be multiplied by the $2x$. It is important to put in brackets if you want the same thing to happen to a whole collection of stuff, both because it tells you that that is what you are doing, and also because it tells anyone else reading your working that that is what you meant. *Many mistakes come from left-out brackets*.

Here is another example of how you need brackets to show that you want different results.

If $a = 2$ then $3a^2 = 3 \times 2 \times 2 = 12$ but $(3a)^2 = 6^2 = 36$. The brackets are necessary to show that it is the whole of $3a$ which is to be squared.

EXERCISE 1.A.1

Try these questions yourself now.

(1) Put the following together as much as possible.

(a) $3a + 2b + 5a + 7c - b - 4c$ (b) $3ab + b + 5a + 2b + 2ba$

(c) $7p + 3pq - 2p + 2pq + 8q$ (d) $5x + 2y - 3x + xy + 3y + 2xy$

(2) If $a = 2$ and $b = 1$, find

(a) a^3 (b) $5a^2$ (c) $(5a)^2$ (d) b^2 (e) $2a^2 + 3b^2$

(3) Multiply the following together.

(a) $(2x)(3y)$ (b) $(3x^2)(5xy)$ (c) $3(2a + 3b)$ (d) $2a(3a + 5b)$
(e) $2p(3p^2 + 2pq + q^2)$ (f) $2x^2(3x + 2xy + y^2)$

1.A.(d) Working out in the right order

If you are replacing letters by numbers, then you must stick to the following rules to work out the answer from these numbers.

(1) In general, we work from left to right.
(2) Any working inside a bracket must be done first.
(3) When doing the working out, first find any powers, then do any multiplying and dividing, and finally do any adding and subtracting.

Here are two examples.

EXAMPLE (1) If $a = 2$, $b = 3$, $c = 4$ and $d = 6$, find $3a(2d + bc) - 4c$.

- Find the inside of the bracket, which is $2 \times 6 + 3 \times 4 = 12 + 12 = 24$.
- Multiply this by $3a$, giving $6 \times 24 = 144$.
- Find $4c$, which is $4 \times 4 = 16$.
- Finally, we have $144 - 16 = 128$.

EXAMPLE (2) If $x = 2$, $y = 3$, $z = 4$ and $w = 6$, work out the value of $x(2y^2 - z) + 3w^2$.

We start by working out the inside of the bracket.

- Find y^2 which is 9.
- The bracket comes to $2 \times 9 - 4 = 14$.
- Multiply this by x, getting 28.
- $w^2 = 6^2 = 36$ so $3w^2 = 108$.
- Finally, we get $28 + 108 = 136$.

EXERCISE 1.A.2

Now try the following yourself.

(1) If $a = 2$, $b = 3$, $c = 4$, $d = 5$ and $e = 0$ find the values of:

(a) $ab + cd$ (b) ab^2e (c) ab^2d (d) $(abd)^2$ (e) $a(b + cd)$
(f) $ab^2d + c^3$ (g) $ab + d - c$ (h) $a(b + d) - c$

(2) Multiply out the following, tidying up the answers by putting together as much as possible.

(a) $3x(2x + 3y) + 4y(x + 7y)$ (b) $5p^2(2p + 3q) + q^2(3p + 5q) + pq(p + 2q)$

Check your answers to these two questions, before going on.

Questions (3) and (4) are very similar to (1) and (2) and will give you some more practice if you need it.

(3) If $a = 3$, $b = 4$, $c = 1$, $d = 5$ and $e = 0$ find the values of:

(a) a^2 (b) $3b^2$ (c) $(3b)^2$ (d) c^2 (e) $ab + c$ (f) $bd - ac$ (g) $b(d - ac)$
(h) $d^2 - b^2$ (i) $(d - b)(d + b)$ (j) $d^2 + b^2$ (k) $(d + b)(d + b)$
(l) $a^2b + c^2d$ (m) $5e(a^2 - 3b^2)$ (n) $a^b + d^a$

(4) Multiply out and collect like terms together if possible:

(a) $3a(2b + 3c) + 2a(b + 5c)$ (b) $2xy(3x^2 + 2xy + y^2)$

(c) $5p(2p + 3q) + 2q(3p + q)$ (d) $2c^2(3c + 2d) + 5d^2(2c + d)$

1.A.(e) Using negative numbers

We shall need to be able to do more complicated things with minus signs than we have met so far, so here is a reminder about dealing with signed numbers.

Ordinary numbers, such as 6, are written as +6 in order to show that they are different from negative numbers such as –5. If the sign in front of a number is +, then it can sometimes be left out. (We don't speak of having +2 apples, for example.) A negative sign can never be left out, in any working combination of numbers.

One way of understanding how signed numbers work is to think of them in terms of money. Then +2 represents having £2, and –3 represents owing £3, etc.

So using brackets to keep each number and its sign conveniently connected, we have for example:

$(+2) + (+5) = (+7)$	Ordinary addition.
$(-3) + (-7) = (-10)$	Adding two debts.
$(+4) + (-9) = (-5)$	You still have a debt.
$(+3) - (-7) = (+10)$	Taking away a debt means you gain.

The same idea carries through to multiplication (which can be thought of as repeated addition, so 3×2 means 3 lots of 2, or adding 2 to itself three times).

Some examples are:

$(+2) \times (-3) = (-6)$	Doubling a debt!
$(-3) \times (+5) = (-15)$	Taking away 3 lots of 5.
$(-3) \times (-7) = (+21)$	Taking away a debt of 7 three times.

The rule for multiplying signed numbers

Two signs which are the same give plus and two different signs give minus.

Here are two examples of this in action.

(1) $3a - 2(b - 2a) + 7b = 3a - 2b + 4a + 7b = 7a + 5b.$

(2) $2p - (p + 2q - m).$

Here, you can think of the minus sign outside the bracket as meaning –1, so that when the bracket is multiplied by it, all the signs inside it will change.

We get $2p - p - 2q + m = p - 2q + m.$

EXERCISE 1.A.3

Now try the following questions.

Multiply out the following, tidying up the answers as much as possible.

(1) $2x - (x - 2y) + 5y$ (2) $4(3a - 2b) - 6(2a - b)$

(3) $6(2c + d) - 2(3c - d) + 5$ (4) $6a - 2(3a - 5b) - (a + 4b)$

(5) $3x(2x - 3y + 2z) - 4x(2x + 5y - 3z)$ (6) $2xy(3x - 4y) - 5xy(2x - y)$

(7) $2a^2(3a - 2ab) - 5ab(2a^2 - 4ab)$ (8) $-3p - (p + q) + 2q(p - 3)$

1.A.(f) Putting into brackets, or factorising

The process described in the previous section can be done in reverse, so, for example, $xy + xz = x(y + z)$.

This reverse process is called **factorisation** and x is called a **factor** of the expression, that is, something you multiply by to get the whole answer, just as 2, 3, 4, 6 are all **factors** of 12. We can say $12 = 3 \times 4 = 2 \times 6$. Each factor divides into 12 exactly.

Here are three examples showing this process happening.

(1) $3a^2 + 2ab = a(3a + 2b)$. This is as far as we can go.
(2) $3p^2q + 4pq^2 = pq\,(3p + 4q)$ factorising as much as possible.
(3) $4a^2b^3 - 6a^3b^2 = 2a^2b^2(2b - 3a)$ factorising as far as possible.

! $xy + x = x(y + 1)$ *not* $x(y + 0)$ because $x \times 1 = x$ but $x \times 0 = 0$.

HELPFUL HINT It is useful to remember that factorisation is just the reverse process to multiplying out. If you are at all doubtful that you have factorised correctly, you can check by multiplying out your answer that you do get back to what you started with originally.

Here's an example.

If you factorise $3c^2 + 2cd + c$, which of the following gives the right answer?

(1) $3c(c + 2d + 1)$ (2) $c(3c + 2d)$ (3) $c(3c + 2d + 1)$.

Multiplying out gives (1) $3c^2 + 6cd + 3c$ (2) $3c^2 + 2cd$ and (3) $3c^2 + 2cd + c$ so (3) is the correct one.

EXERCISE 1.A.4

Factorise the following yourself, taking out as many factors as you can.

(1) $5a + 10b$ (2) $3a^2 + 2ab$ (3) $3a^2 - 6ab$
(4) $5xy + 8xz$ (5) $5xy - 10xz$ (6) $a^2b + 3ab^2$
(7) $4pq^2 - 6p^2q$ (8) $3x^2y^3 + 5x^3y^2$
(9) $4p^2q + 2pq^2 - 6p^2q^2$ (10) $2a^2b^3 + 3a^3b^2 - 6a^2b^2$

1.B Multiplications and factorising: the next stage

1.B.(a) Self-test 2

This section also starts with a self-test. It is sensible to do it even if you think you don't have any problems with these because it won't take you very long to check that you are in this happy state. It's a good idea to cover my answers until you've done yours.

(A) Multiply out the following
(1) $(2x + 3y)(x + 5y)$ (2) $(3a - 5b)(2a - b)$ (3) $(3x + 2)^2$
(4) $(2y - 5)^2$ (5) $(2p^2 + 3pq)(q^2 - 2pq)$

Factorise the following.

(B)	(1) $x^2 + 9x + 14$	(2) $y^2 + 8y + 12$	(3) $x^2 + 8x + 16$	(4) $p^2 + 13p + 22$
(C)	(1) $2x^2 + 7x + 3$	(2) $3a^2 + 16a + 5$	(3) $3b^2 + 10b + 7$	(4) $5x^2 + 8x + 3$
(D)	(1) $x^2 + x - 2$	(2) $2a^2 + a - 15$	(3) $2x^2 + 5x - 12$	(4) $p^2 - q^2$
	(5) $6y^2 - 19y + 10$	(6) $4x^2 - 81y^2$	(7) $6x^2 - 19x + 10$	(8) $4x^2 - 12x + 9$

As in the first test, give yourself one point for each correct answer so that the highest total score is 30. Again, if you got 25 or less, work through this following section.

If you are in any doubt, it is much better to get it sorted out now, because lots of later work will depend on it.

These are the answers that you should have.

(A)	(1) $2x^2+13xy+15y^2$	(2) $6a^2-13ab+5b^2$	(3) $9x^2+12x+4$	
	(4) $4y^2-20y+25$	(5) $3pq^3-4p^3q-4p^2q^2$		
(B)	(1) $(x+2)(x+7)$	(2) $(y+2)(y+6)$	(3) $(x+4)^2$	(4) $(p+2)(p+11)$
(C)	(1) $(2x+1)(x+3)$	(2) $(3a+1)(a+5)$	(3) $(3b+7)(b+1)$	(4) $(5x+3)(x+1)$
(D)	(1) $(x+2)(x-1)$	(2) $(2a-5)(a+3)$	(3) $(2x-3)(x+4)$	(4) $(p-q)(p+q)$
	(5) $(3y-2)(2y-5)$	(6) $(2x-9y)(2x+9y)$	(7) $(3x-2)(2x-5)$	(8) $(2x-3)^2$

1.B.(b) Multiplying out two brackets

To multiply out two brackets, each bit of the first bracket must be multiplied by each bit of the second bracket, so

$$\boxed{(a + b)(c + d) = ac + bd + ad + bc.}$$

The $ac + bd + ad + bc$ can be written in any order.

You could also think of this process, if you like, as

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

You can see this working numerically by putting $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

$$(a + b)(c + d) = (1 + 2)(3 + 4) = 3 \times 7 = 21$$

and

$$ac + ad + bc + bd = 3 + 4 + 6 + 8 = 21.$$

Also, you can see that the order of doing the multiplying doesn't matter, since

$$ac + bd + bc + ad = 3 + 8 + 6 + 4 = 21 \text{ too.}$$

Figure 1.B.1 shows this process happening with areas. $(a + b)(c + d)$ gives the total area of the rectangle.

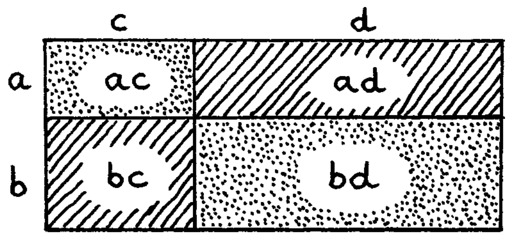

Figure 1.B.1

Exactly the same system is used to work out $(a + b)^2$. We have

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2 = a^2 + 2ab + b^2$$

We can see this working in Figure 1.B.2.

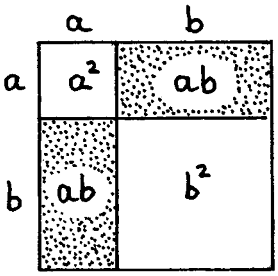

Figure 1.B.2

We can see the two squares and the two same-shaped rectangles.

> Don't forget the middle bit of $2ab$.

The diagram shows that $(a + b)^2$ is not the same thing as $a^2 + b^2$. In a similar way, we have

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2.$$

What happens if the signs are opposite ways round, so we have $(a + b)(a - b)$?

> We get
>
> $$(a + b)(a - b) = a^2 - b^2$$
>
> because the middle bits cancel out.
>
> This result is called **the difference of two squares**.

You need to be good at spotting examples of this because it is of very great importance in simplifying and factorising in many different situations.

To help you to get good at this, here are some further examples.

Put back into two brackets (1) $x^2 - 9y^2$, (2) $49a^2 - 64b^2$.

The answers are (1) $(x + 3y)(x - 3y)$ and (2) $(7a + 8b)(7a - 8b)$.

Check these are true by multiplying them back out, and then try the following ones for yourself.

(1) $x^2 - y^2$ (2) $4a^2 - 9b^2$ (3) $16p^2 - 9q^2$ (4) $16a^2 - 25b^2$ (5) $36p^2 - 100q^2$

These are the answers that you should have.

(1) $(x + y)(x - y)$ (2) $(2a + 3b)(2a - 3b)$ (3) $(4p + 3q)(4p - 3q)$
(4) $(4a + 5b)(4a - 5b)$ (5) $(6p + 10q)(6p - 10q)$

In each case, the brackets can equally well be written the other way round since the letters are standing for numbers.

Here is a more complicated example of multiplication of brackets.

$$(3x + xy)(xy + y^2) = 3x^2y + x^2y^2 + 3xy^2 + xy^3$$

Again, the basic strategy is the same. Each bit or chunk of the first bracket is multiplied by each bit or chunk of the second one.

(This can be checked by putting $x = 2$ and $y = 3$. Each side should come to 180.)

EXERCISE 1.B.1

Multiply out the following pairs of brackets.

(1) $(x + 2)(x + 3)$ (2) $(a + 3)(a - 4)$ (3) $(x - 2)(x - 3)$
(4) $(p + 3)(2p + 1)$ (5) $(3x - 2)(3x + 2)$ (6) $(2x - 3y)(x + 2y)$
(7) $(3a - 2b)(2a - 5b)$ (8) $(3x + 4y)^2$ (9) $(3x - 4y)^2$
(10) $(3x + 4y)(3x - 4y)$ (11) $(2p^2 + 3pq)(5p + 3q)$ (12) $(2ab - b^2)(a^2 - 3ab)$
(13) $(a + b)(a^2 - ab + b^2)$ (14) $(a - b)(a^2 + ab + b^2)$
(15) Try working through the following steps.
- (a) Think of a positive whole number, and write down its square.
- (b) Add 1 to your original whole number, and multiply the result by the original number with 1 taken away from it.
- (c) Repeat this process twice more.
- (d) Describe in words what seems to be happening.
- (e) Must this always happen whatever your starting number is?

Show that it must by taking a starting number of n so that you can see exactly what must happen every time.

1.B.(c) More factorisation: putting things back into brackets

Again, the reverse process to multiplying out two brackets is called **factorisation**. Very often it is important to be able to replace a more complicated expression by two simpler expressions multiplied together.

We have already done some examples of this, when we were working with the difference of two squares in the previous section.

What happens, though, if there is a middle bit to be sorted out?

For example, suppose we have $x^2 + 7x + 12$.

Can we replace this expression by two multiplied brackets?

We would have $x^2 + 7x + 12 =$ (*something*) (*something*), and we have to find out what the somethings must be.

We can see that we will need to have x at the beginning of each of the brackets.

Both signs in the brackets are positive since the left-hand side is all positive, so at the ends we need two numbers which when multiplied give +12 and which when added give +7. What two numbers will do this?

+3 and +4 will do what we want, so we can say $x^2 + 7x + 12 = (x + 3)\,(x + 4)$, giving us an alternative way of writing this expression.

Equally, $x^2 + 7x + 12 = (x + 4)(x + 3)$.

The order of the brackets is not important because multiplication of numbers gives the same answer either way on. For example, $2 \times 3 = 3 \times 2 = 6$.

In all the questions which follow, your answer will be equally correct if you have your brackets in the opposite order from mine.

EXERCISE 1.B.2

Try putting the following into brackets yourself.

(1) $x^2 + 8x + 7$ (2) $p^2 + 6p + 5$ (3) $x^2 + 7x + 6$

(4) $x^2 + 5x + 6$ (5) $y^2 + 6y + 9$ (6) $x^2 + 6x + 8$

(7) $a^2 + 7a + 10$ (8) $x^2 + 9x + 20$ (9) $x^2 + 13x + 36$

Now, a step further! Suppose we have $2x^2 + 7x + 3 =$ (*something*) (*something*). This time we need $2x$ and x at the fronts of the brackets to give the $2x^2$. If it is possible to factorise this with whole numbers then the ends will need 1 and 3 to give $1 \times 3 = 3$.

Do we need $(2x + 3)(x + 1)$ or $(2x + 1)(x + 3)$?

Multiplying out, we see that

$(2x + 3)(x + 1) = 2x^2 + 5x + 3$ which is wrong,
$(2x + 1)(x + 3) = 2x^2 + 7x + 3$ so this is the one we need.

EXERCISE 1.B.3

Try factorising these for yourself now.

(1) $3x^2 + 8x + 5$ (2) $2y^2 + 15y + 7$ (3) $3a^2 + 11a + 6$

(4) $3x^2 + 19x + 6$ (5) $5p^2 + 23p + 12$ (6) $5x^2 + 16x + 12$

The system is exactly the same if the expression involves minus signs. Here are two examples showing what can happen.

EXAMPLE (1) Factorise $x^2 - 10x + 16$.

Here we require two numbers which when multiplied give +16, and which when put together give –10. Can you see what they will be?

Both the numbers must be negative, and we see that –2 and –8 will fit the requirements. This gives us $x^2 - 10x + 16 = (x - 2)(x - 8) = (x - 8)(x - 2)$.

EXAMPLE (2) Factorise $x^2 - 3x - 10$.

Now we require two numbers which when multiplied give –10 and which when put together give –3. Can you see what we will need?

This time, to give the –10, they need to be of different signs.
We see that –5 and +2 will do what we want, so we have

$$x^2 - 3x - 10 = (x - 5)(x + 2) = (x + 2)(x - 5).$$

Remember that it makes no difference which way round you write the brackets.

EXERCISE 1.B.4

Now try factorising the following yourself.

(1) $x^2 - 11x + 24$ (2) $y^2 - 9y + 18$ (3) $x^2 - 11x + 18$

(4) $p^2 + 5p - 24$ (5) $x^2 + 4x - 12$ (6) $2q^2 - 5q - 3$

(7) $3x^2 - 10x - 8$ (8) $2a^2 - 3a - 5$ (9) $2x^2 - 5x - 12$

(10) $3b^2 - 20b + 12$ (11) $9x^2 - 25y^2$ (12) $16x^4 - 81y^4$, a sneaky one!

1.C Using fractions

Very many students find handling fractions in algebra quite difficult, but it is important to be able to simplify these fractions as far as possible. This is because they often come into longer pieces of working and, if you do not simplify as you go along, the whole thing will become hideously complicated. It is only too likely then that you will make mistakes.

This section is designed to save you from this. You will find that if you understand how arithmetical fractions work then using fractions in algebra will be easy. If you have been using a calculator to do fractions, it's likely that you will have forgotten how they actually work, so I've drawn some little pictures of what is happening to help you.

If you think that you can already work well with fractions, try some of each exercise to be sure that there are no problems before you move on to the next section.

Because we are looking here at what we can and can't do with fractions, we shall need to use the sign .

> The sign means 'is not equal to'.

1.C.(a) Equivalent fractions and cancelling down

> $\dfrac{a}{b}$ means a divided by b.
>
> a is called the **numerator** and b is called the **denominator**.

In dividing, the order that the letters are written in matters, unlike $a \times b$, which is the same as $b \times a$.

The order also matters with subtraction; $a - b$ is not the same as $b - a$ unless both a and b are zero. But $a + b = b + a$ always.

For example, $2 \times 3 = 3 \times 2$ and $2 + 3 = 3 + 2$, but $\frac{2}{3} \quad \frac{3}{2}$ and $2 - 3 \quad 3 - 2$.

$$\text{Also,}\quad \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}. \quad \text{For example,}\quad \frac{2+3}{7} = \frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

The whole of $a + b$ is divided by c, and so we can get the same result by splitting this up into two separate divisions. The line in the fraction is effectively working as a bracket.

In fact, it is safer to write $\dfrac{a+b}{c}$ as $\dfrac{(a+b)}{c}$ if it is part of some working.

In $\dfrac{a}{b+c}$, the number a is divided by the whole of the number $(b + c)$.

From this, we see that

$$\frac{a}{b+c} \quad \frac{a}{b} + \frac{a}{c}.$$

You can check this by putting $a = 4$, $b = 2$, $c = 3$, say.

Dividing by c is the same as multiplying by $1/c$, so

$$\frac{a+b}{c} = \frac{1}{c}(a+b).$$

For example, if $a = 6$, $b = 4$, and $c = 2$ then

$$\frac{6+4}{2} = \tfrac{1}{2}(6+4) = 5.$$

If you find half of 10, it is the same as dividing 10 by 2.

Fractions always keep the same value if they are multiplied or divided top and bottom by the same number, so

$$\frac{4}{6} = \frac{8}{12} = \frac{6}{9} = \frac{2}{3}, \text{ etc.}$$

These are shown in the drawings in Figure 1.C.1.

These four equal fractions are said to be **equivalent** to each other. The process of dividing the top and bottom of a fraction by the same number is called **cancellation** or cancelling down.

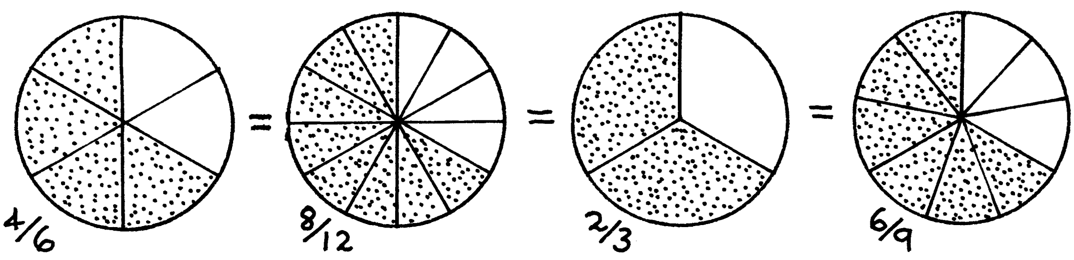

Figure 1.C.1

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \quad \text{not} \quad \frac{ab}{ac}.$$

For example, $4\left(\frac{2}{3}\right) = \frac{4 \times 2}{3}$ not $\frac{4 \times 2}{4 \times 3}$ which is still $\frac{2}{3}$.

In words, four lots of two thirds is eight thirds.

This works in exactly the same way with fractions in algebra.

So, for example:

$$\frac{2a}{5a} = \frac{2}{5} \quad \text{(dividing top and bottom by } a\text{)}$$

$$\frac{xw}{yw} = \frac{x}{y} \quad \text{(dividing top and bottom by } w\text{)}$$

and $$\frac{2a^3b}{a^2b^2} = \frac{2a}{b} \quad \text{(dividing top and bottom by } a^2b\text{).}$$

Check these three results by giving your own values to the letters.

When doing this, it is important to avoid values which would involve you in trying to divide by zero, because this cannot be done.

You can use a calculator to investigate this by dividing 4, say, by a very small number, say 0.00001.

Now repeat the process, dividing 4 by an even smaller number.

The closer the number you divide by gets to zero, the larger the answer becomes. In fact, by choosing a sufficiently small number, you can make the answer as large as you please.

If you try to divide by zero itself, you get an ERROR message.

EXERCISE 1.C.1

Cancel down the following fractions yourself as far as possible.

(1) $\frac{9}{12}$ (2) $\frac{6}{30}$ (3) $\frac{25}{95}$ (4) $\frac{24}{64}$ (5) $\frac{5x}{8x}$ (6) $\frac{ab}{ac}$

(7) $\frac{3y^2}{2y}$ (8) $\frac{8pq}{2q}$ (9) $\frac{4a^2}{2ab}$ (10) $\frac{3x^2y^3}{2xy^4}$ (11) $\frac{6p^2q}{5pq^2}$ (12) $\frac{5ab}{b^3}$

1.C.(b) Tidying up more complicated fractions

Sometimes, the process of factorising will be very important in simplifying fractions. Here are some examples of possible simplifications, and some warnings of what *can't* be done. If you have always found this sort of thing difficult, it may help you here to highlight the matching parts which are cancelling with each other in the same colour.

(1) $\dfrac{xy + xz}{xw} = \dfrac{x(y + z)}{xw} = \dfrac{y + z}{w}$

dividing top and bottom by x.

(2) $\dfrac{ab + ac}{b + c} = \dfrac{a(b + c)}{b + c} = a$

dividing top and bottom by the whole chunk of $(b + c)$.

(3) $\dfrac{ab + c}{b + c}$ can't be simplified.

We can't cancel the $(b + c)$ here because a only multiplies b.

(4) $\dfrac{x + xy}{x^2} = \dfrac{x(1 + y)}{x^2} = \dfrac{1 + y}{x}$

dividing top and bottom by x.

(5) $\dfrac{x^2 + 5x + 6}{x^2 - 3x - 10} = \dfrac{(x + 3)(x + 2)}{(x - 5)(x + 2)} = \dfrac{x + 3}{x - 5}$

dividing top and bottom by $(x + 2)$.

(6) $\dfrac{x^2(x^2 + xy)}{x} = x(x^2 + xy)$

dividing top and bottom by x.

! It is not true that $\dfrac{x(x^2 + xy)}{x} = x + y$.

This wrong answer comes from cancelling the x twice on the top of the fraction, but only once underneath.

It is like saying $\frac{1}{2}(4)(6) = (2)(3) = 6$ but really $\frac{1}{2}(4)(6) = \frac{1}{2}(24) = 12$.

You can halve either the 4 or the 6 but not both!

! (7) $\dfrac{xy + z}{xw}$ is *not* the same as $\dfrac{y + z}{w}$.

We cannot cancel the x here because x is only a factor of *part* of the top. You can check this by putting $x = 2$, $y = 3$, $z = 4$, and $w = 5$. Then

$$\frac{xy + z}{xw} = \frac{10}{10} = 1 \quad \text{and} \quad \frac{y + z}{w} = \frac{7}{5}$$

DELICATE POINT

If we had put $x = 1$, the difference would not have shown up, since both answers would have been $\frac{7}{5}$.

This is because multiplying by 1 actually leaves numbers unchanged.

This example shows that checking with numbers is only a check, and never a proof that something is true.

EXERCISE 1.C.2

Try these questions yourself now.

(1) Which of the following fractions are the same as each other (equivalent)?

(a) $\frac{2}{3}, \frac{4}{9}, \frac{12}{18}, \frac{10}{15}, \frac{2}{6}, \frac{6}{9}$ (b) $\frac{ax}{bx}, \frac{a}{b}, \frac{a(c+d)}{b(c+d)}, \frac{a^2x}{abx}$

(c) $\frac{ab+ac}{ad}, \frac{ab+c}{ad}, \frac{b+c}{d}$ (d) $\frac{x}{x+y}, \frac{xz}{xz+yz}, \frac{xp}{x+yp}$

(2) Factorise and cancel down the following fractions if possible.

(a) $\frac{2x+6y}{6x-8y}$ (b) $\frac{6a-9b}{4a-6b}$ (c) $\frac{px-pq}{p^2-px}$

(d) $\frac{3x+2y}{6x}$ (e) $\frac{2xy+5xz}{6x}$ (f) $\frac{4xz+6yz}{2x+3y}$

(g) $\frac{2p-3q}{2p+3q}$ (h) $\frac{x^2-y^2}{(x+y)^2}$ (i) $\frac{x^2+5x+6}{x^2+x-2}$

1.C.(c) Adding fractions in arithmetic and algebra

It is particularly easy to add fractions which have the same number underneath.

For example, $\frac{2}{7} + \frac{3}{7} = \frac{5}{7}$. I've drawn this one in Figure 1.C.2 below.

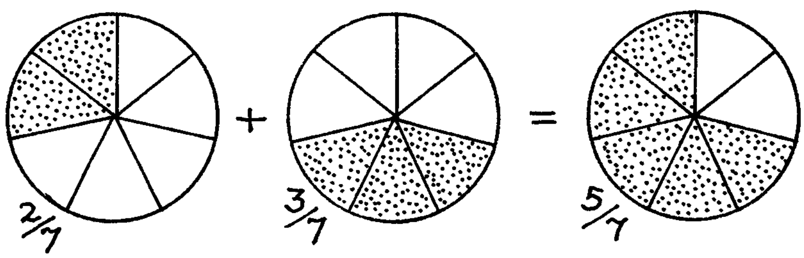

Figure 1.C.2

If the fractions which we want to add don't have the same denominator then we have to first rewrite them as equivalent fractions which do share the same denominator.

For example, to find $\frac{2}{3} + \frac{3}{4}$ we use $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$.

The two fractions have both been written as parts of 12. The number 12 is called the **common denominator**. It's now very easy to add them, and we have

$$\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{17}{12}.$$

The answer of $\frac{17}{12}$ can also be written as $1\frac{5}{12}$, but in general, for scientific and engineering purposes, it is better to leave such arithmetical fractions in their top-heavy state.

You should be safe now from the most usual mistake made when adding fractions, which is to add the tops and add the bottoms.

> $\frac{1}{6}+\frac{3}{4}$ (for example) is *not* $\frac{1+3}{6+4}=\frac{4}{10}$.

We can see that this must be wrong from Figure 1.C.3.

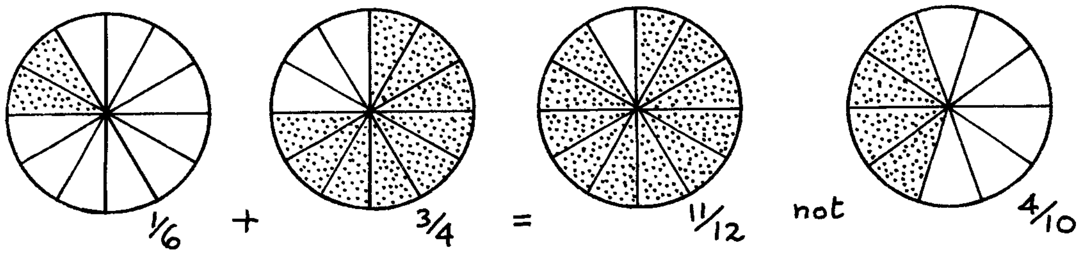

Figure 1.C.3

EXERCISE 1.C.3

Since the process in arithmetic is exactly the same as the process we use to add fractions in algebra, it is worth practising adding some numerical fractions yourself without using a calculator, before we move on to this.

Try adding these three.

(1) $\frac{3}{4}+\frac{2}{7}$ (2) $\frac{2}{3}+\frac{4}{5}$ (3) $\frac{1}{2}+\frac{2}{3}+\frac{4}{5}$

The letters work in exactly the same way as the numbers. We can say

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

where a, b, c and d are standing for unknown numbers, and neither b nor d are zero. We have written both fractions as parts of bd to make it easy to add them.

Indeed, we can say

$$\frac{A}{B}+\frac{C}{D}=\frac{AD}{BD}+\frac{BC}{BD}=\frac{AD+BC}{BD}$$

where A, B, C and D are standing for whole lumps or chunks of letters and numbers.

As an example of this, we will find

$$\frac{x+2y}{x-y}+\frac{3x+2y}{x+3y}.$$

Here, $A = x + 2y$, $B = x - y$, $C = 3x + 2y$ and $D = x + 3y$. So we have:

$$\frac{(x + 2y)(x + 3y)}{(x - y)(x + 3y)} + \frac{(3x + 2y)(x - y}{(x + 3y)(x - y)} = \frac{(x + 2y)(x + 3y) + (3x + 2y)(x - y)}{(x - y)(x + 3y)}$$

$$= \frac{x^2 + 5xy + 6y^2 + 3x^2 - xy - 2y^2}{(x - y)(x + 3y)}$$

$$= \frac{4x^2 + 4xy + 4y^2}{(x - y)(x + 3y)} = \frac{4(x^2 + xy + y^2)}{(x - y)(x + 3y)}.$$

We don't usually multiply out the brackets on the bottom, because then we might miss a possible cancellation. (This saves you some work.)

Try combining $\dfrac{3x - 2}{x + 3} + \dfrac{2x - 3}{x + 1}$ into a single fraction, yourself.

The working should go as follows:

$$\frac{(3x - 2)(x + 1)}{(x + 3)(x + 1)} + \frac{(2x - 3)(x + 3)}{(x + 1)(x + 3)} = \frac{(3x - 2)(x + 1) + (2x - 3)(x + 3)}{(x + 3)(x + 1)}$$

$$= \frac{3x^2 + x - 2 + 2x^2 + 3x - 9}{(x + 3)(x + 1)}$$

$$= \frac{5x^2 + 4x - 11}{(x + 3)(x + 1)}.$$

(Remember that the order in which we multiply the brackets doesn't matter.)

1.C.(d) Repeated factors in adding fractions

Sometimes, the addition is a little easier because there is a repeated factor. Here's a numerical example of this.

$\dfrac{3}{4} + \dfrac{5}{6}$ has a repeated factor of 2 underneath.

So, instead of saying

$$\frac{3}{4} + \frac{5}{6} = \frac{18}{24} + \frac{20}{24} = \frac{38}{24} = \frac{19}{12}$$

we can say more directly

$$\frac{3}{4} + \frac{5}{6} = \frac{9}{12} + \frac{10}{12} = \frac{19}{12}.$$

The number 12, which is the smallest number which both 4 and 6 will divide into, is called the **lowest common denominator** or **l.c.d.** for short.

This same simplification applies to fractions in algebra.