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Homotopy fixed points in the algebraic K-theory of certain infinite discrete groups

Gunnar Carlsson*

1 Introduction

Since their introduction 15 years ago, Quillen's algebraic K-groups of rings have remained fairly difficult to compute in most cases of interest. This paper is a brief sketch of an attempt to remedy this situation in the case of group rings of certain discrete groups. Group rings are of particular interest since the algebraic K-groups are in this case related to the geometry of manifolds with the given group as fundamental group. Specifically, the algebraic K-groups of $\mathbf{Z}(\pi_1 M)$ are involved in the description of the space of self-homeomorphisms of M.

Recall that the zero-th space of Quillen's K-theory spectrum associated to a ring A is the space $BGL^+(A)$ [7]. One obvious map into the space $BGL^+(\mathbf{Z}\Gamma)$ arises from the existence of a tensor product map $BGL_n(A) \times BGL_m(B) \xrightarrow{\text{“}\otimes\text{”}} BGL_{n\cdot m}(A \otimes_C B)$, where A and B are C-algebras, as follows. We note that the group Γ is contained in the group $GL_1(\mathbf{Z}\Gamma)$, as one-by-one matrices with entry the given element of Γ. Thus, we have a map $\Gamma \times GL_n\mathbf{Z} \xrightarrow{\text{“}\otimes\text{”}} GL_n\mathbf{Z}\Gamma$. Applying the classifying space functor and passing to the limit over n we obtain a map $B\Gamma \times BGL^+\mathbf{Z} \to BGL^+(\mathbf{Z}\Gamma)$. After suitable interpretation, one finds that this map can be delooped so as to obtain a map of spectra $B\Gamma_+ \wedge \underset{\sim}{K}\mathbf{Z} \xrightarrow{\alpha} \underset{\sim}{K}\mathbf{Z}\Gamma$, where $\underset{\sim}{K}$ denotes the K-theory spectrum, and α is called the assembly map. In many cases, one conjectures that the map is a split injection onto a wedge summand, or perhaps that it is an equivalence. Much work has been done in this direction.

(a) Waldhausen [10] showed the map to be an equivalence in many cases, and analysed the failure of the map to be an equivalence in many others. He studied groups constructed from infinite cyclic ones by amalgamated sum and Laurent extension procedures.

(b) Quinn showed that if Γ is the fundamental group of a flat manifold, then the assembly map is an equivalence after rationalization [8].

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(c) The L-theory analogue of the conjecture that α is injective on a wedge summand is a strong form of the Novikov conjecture. The actual Novikov conjecture is that α is an injection after rationalization. The K-theory version has been proved for groups with finitely generated homology by Bökstedt, Hsiang, and Madsen.

(d) The A-theory analogue of α has been studied with spectacular success by Farrell and Jones [5]. A-theory means Waldhausen's K-theory of the spaces applied to $B\Gamma$. Farrell and Jones completely describe $A(B\Gamma)$, when Γ is the fundamental group of a negatively curved manifold. A corollary of their work is that the ordinary K-theory assembly map is an equivalence after rationalization for these groups.

2 A homotopy-theoretic approach

We'll describe an approach to this problem fundamentally different from those used in the above pieces of work. To motivate the procedures, we'll consider first the case of the complex group ring of a finite group. Here, Atiyah [1] and Atiyah-Segal [2] observed that the K-theory spectrum could be viewed as the fixed point spectrum of the group G acting on a G-equivariant spectrum having the homotopy type of $\underset{\sim}{K}\mathbf{C}$, which we shall call $K_G(\mathbf{C})$. The zero-th space of $K_G(\mathbf{C})$ is the space BU, equipped with an action arising from all complex representations of G. Now $BU^G \cong \coprod_\rho BU$, where ρ ranges over all virtual representations of G. Atiyah [1] went on to show that if we consider the map $BU^G \to BU^{hG}$, where BU^{hG} denotes the homotopy fixed point set, we often detect the entire space BU^G, and hence the K-theory spectrum of $\mathbf{C}[G]$. See [3] for a discussion of homotopy fixed point sets. Specifically, if G is a p-group Atiyah showed that the map $BU^G \to BU^{hG}$ is an equivalence after p-adic completion. This suggests that we attempt to mimic this procedure, even in the case of infinite groups. We arrive immediately at a problem, since a key point in the Atiyah-Segal program was that G acted on *finite* matrices. When we naively generalize this part of the program, we find that $\mathbf{Z}\Gamma$ is indeed realizable as the fixed point subring of an action of Γ on a ring of matrices, but that the matrices are necessarily infinite. It is easy to show that when one considers the ring of all infinite matrices with entries in $\mathbf{Z}$, whose rows and columns are finite, the associated K-theory spectrum is contractible, and hence so is its homotopy fixed point set. Further, the natural guess for a spectral sequence converging to $K_*\mathbf{Z}\Gamma$ would have $E_2^{p,q}$-term $H^{-p}(\Gamma; K_q\mathbf{Z})$, and many negative-dimensional groups would arise. The second observation shows that this kind of naive approach will not lead very far.

Our new approach can be summarized as follows. Roughly speaking, instead of considering all finite matrices, one should consider an appropriately chosen subring. In the case of $\Gamma = \mathbf{Z}$, one will consider a subring of the ring of all endomorphisms of the free $\mathbf{Z}$-module with basis $\{e_n\}_{n=-\infty}^{\infty}$. The subring in question consists of all those matrices M for which there is a number $N(M)$ so that $M_{ij} = 0$ if $|i-j| > N(M)$. Here, M_{ij} denotes the matrix entry associated to i and j. This subring will be called the ring of *bounded* endomorphisms of the given module, and will be denoted by $\mathcal{B}$. Now, Γ acts on the basis $\{e_n\}$ by $\sigma e_n = e_{n+1}$, where σ is a generator for Γ. This action gives a conjugation action on $\mathcal{B}$, and the fixed point subring is $\mathbf{Z}(\Gamma)$. The important point is that the K-theory of $\mathcal{B}$ is not trivial, but in fact $K_i(\mathcal{B}) = K_{i-1}(\mathbf{Z})$, and $\underset{\sim}{K}\mathcal{B} \cong \Sigma \underset{\sim}{K}\mathbf{Z}$. When we take the homotopy fixed point set $\underset{\sim}{K}\mathcal{B}^{h\Gamma}$, we find that $\underset{\sim}{K}\mathcal{B}^{h\Gamma} \cong \underset{\sim}{K}\mathbf{Z} \vee \Sigma \underset{\sim}{K}\mathbf{Z}$, which is known to be the correct answer for $\underset{\sim}{K}\mathbf{Z}(\Gamma)$ from the localization sequence and homotopy axiom for algebraic K-theory [7]. This suggests very strongly that we are on the right track, and should attempt to find a suitable replacement for $\mathcal{B}$ for a more general class of groups. The suitable framework is the bounded algebraic K-theory of E. Pedersen and C. Weibel [6]. They associate to any metric space X and ring R a spectrum $\underset{\sim}{K}(R;X)$, the K-theory of R with labels in X. This is done as follows. To R and X, we first associate a category $\mathcal{C}_X(R)$, whose objects are free R-modules equipped with a basis $B = \{e_\alpha\}_{\alpha\in A}$ and a "labelling" function $\varphi: B \to X$. The modules may be infinitely generated. A morphism from $\{M, B_M, \varphi_M\}$ to $\{N, B_N, \varphi_N\}$ is an R-linear isomorphism f which is bounded in the following sense. Given a linear transformation $T: M \to N$, let $\{T_{\alpha\beta}\}_{\alpha\in B_M, \beta\in B_N}$ be the matrix of T relative to the bases B_M and B_N. Then f is said to be bounded if there is a number L so that $f_{\alpha\beta} = 0 = (f^{-1})_{\beta\alpha}$ if $d(\varphi_M(\alpha), \varphi_N(\beta)) > L$. The category $\mathcal{C}_X(R)$ is symmetric monoidal, and so is its "idempotent completion" $\widehat{\mathcal{C}}_X(R)$. To any symmetric monoidal category $\mathcal{A}$ one associates a spectrum, as in [9], called Spt$\mathcal{A}$. Now $\underset{\sim}{K}(R,X)$ is defined to be $\mathrm{Spt}(\widehat{\mathcal{C}}_X(R))$.

Pedersen and Weibel proceed to prove several results about their construction. For instance, if E^n denotes $\mathbf{R}^n$ with its standard Euclidean structure, then $\underset{\sim}{K}(R;E^n)_0$ is an n-fold delooping of $\underset{\sim}{K}(R)_0$. The subscript "0" denotes zero-th space. These deloopings are not in general connective; in fact, they are equivalent to the so-called Gersten-Wagner deloopings of $\underset{\sim}{K}(R)_0$. Also, they show that if $X \subseteq S^{n-1}$, and $CX \subseteq E^n$ is the open cone on X, with induced metric, then $\underset{\sim}{K}(R;CX) \cong \Sigma X \wedge \widehat{\underset{\sim}{K}}R$, where $\widehat{\underset{\sim}{K}}R$ is the spectrum whose nth space is $\underset{\sim}{K}(R;E^n)_0$. Finally, it is clear from the construction that if X is a bounded metric space, then $\underset{\sim}{K}(R;X) \cong \widehat{\underset{\sim}{K}}(R)$.

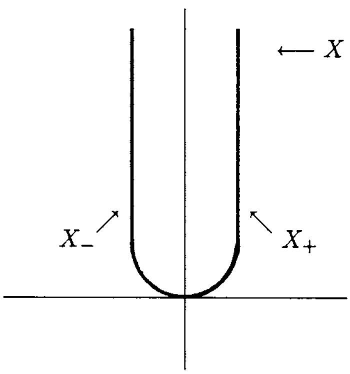
Our generalization of the above construction for $\Gamma = \mathbf{Z}$ now goes as follows. If the metric space X is acted on isometrically by a group Γ, then there is a spectrum $\underset{\sim}{K}_\Gamma(R;X)$, with Γ-action, so that $\underset{\sim}{K}_\Gamma(R;X) \cong \underset{\sim}{K}(R;X)$ non-equivariantly. One can show that if the Γ-action is free, and if X/Γ is bounded, then $\widehat{\underset{\sim}{K}}_\Gamma(R;X)^\Gamma \cong \underset{\sim}{K}(R\Gamma; X/\Gamma) \cong \widehat{\underset{\sim}{K}}(R\Gamma)$. There is an evident induced metric on X/Γ, which is the one we use. We have now achieved our goal of constructing a spectrum with Γ-action whose fixed point spectrum is the K-theory spectrum of $R\Gamma$, if we can find the correct metric space.

Now suppose that $\Gamma = \pi_1 X$, where X is a compact closed manifold. If we equip X with a Riemannian metric, X becomes a bounded metric space. Of course, the Riemannian metric pulls back to the universal cover $\widetilde{X}$, where it is invariant by the action of Γ by deck transformations. Thus $\widetilde{X}$ is a good choice of metric space with free Γ-action. If X were actually flat, then $\widetilde{X} = E^n$, and we would understand $\underset{\sim}{K}(R; E^n)$ from the work of Pedersen and Weibel. In particular, we would find that we have a spectral sequence with $E_2^{p,q} = H^{-p}(\Gamma, K_{q-n}\mathbf{Z})$ converging to $\underset{\sim}{K}_\Gamma(\mathbf{Z}; E^n)^{h\Gamma}$. If we examine the spectral sequence, we find that the groups which appear are precisely those which appear in a similar spectral sequence for $\pi_*(B\Gamma_+ \wedge \underset{\sim}{K}\mathbf{Z})$, because of Poincaré duality in $H^*(\Gamma; \mathbf{Z})$. One can make a precise argument which shows that in fact $\underset{\sim}{K}_\Gamma(\mathbf{Z}; E^n)^{h\Gamma} \cong B\Gamma_+ \wedge \underset{\sim}{K}\mathbf{Z}$, and that the composite $B\Gamma_+ \wedge \underset{\sim}{K}\mathbf{Z} \xrightarrow{\alpha} \underset{\sim}{K}(\mathbf{Z}\Gamma) \to \underset{\sim}{K}_\Gamma(\mathbf{Z}; E^n)^{h\Gamma}$ is an equivalence, allowing us to conclude that α is the inclusion of a wedge factor.

Suppose now $\Gamma = \pi_1 X$, where X is closed, compact, and $\widetilde{X} = \mathbf{R}^n$. In general, the metric will not be flat, and we are unable to use the Pedersen-Weibel results. One can hope, however, to use Mayer-Vietoris techniques to understand $\underset{\sim}{K}(R; \widetilde{X})$ in certain situations. In fact, Pedersen and Weibel used a Mayer-Vietoris sequence in proving their result for Euclidean space. One could ask, then, if a metric space X is decomposed as a union $X = Y \cup Z$, does one have a cofibre sequence

$$\underset{\sim}{K}(R; Y \cap Z) \longrightarrow \underset{\sim}{K}(R; Y) \vee \underset{\sim}{K}(R; Z) \longrightarrow \underset{\sim}{K}(R; X)\,?$$

In general, this fails as one can see from the following picture in the plane E^2.



The subset X is equipped with the induced distance function (not the induced Riemannian metric) from E^2. Of course, X is homeomorphic to the real line, and $X = X_+ \cup X_-$. One shows easily that $\underset{\sim}{K}(R; X_+) \cong * \cong \underset{\sim}{K}(R; X_-)$, and of course $\underset{\sim}{K}(R; X_+ \cap X_-) \cong \underset{\sim}{K}(R)$. Thus, if there were a Mayer-Vietoris sequence, we would have a cofibre sequence

$$\underset{\sim}{K}(R; X_+ \cap X_-) \longrightarrow \underset{\sim}{K}(R; X_+) \vee \underset{\sim}{K}(R; X_-) \longrightarrow \underset{\sim}{K}(R; X),$$

and hence that $\underset{\sim}{K}(R; X) \cong \Sigma \underset{\sim}{K}(R; X_+ \cap X_-)$. On the other hand, it is easy to see that projection on the y-axis induces an equivalence $\underset{\sim}{K}(R; X) \to \underset{\sim}{K}(R; \mathbf{R}_+)$, where $\mathbf{R}_+$ denotes the positive y-axis. Pedersen and Weibel have shown this latter space to be contractible, so the sequence cannot be correct as it stands.

Fortunately, it is not too difficult to remedy this situation. If U is a subset of a metric space X, and r is a number, then let $N_r U = \{x \in X \mid \exists u \in U \text{ with } d(x,u) \leq r\}$. Then it turns out that one can construct a Mayer-Vietoris cofibration sequence

$$\begin{aligned} \varinjlim_r \underset{\sim}{K}(R; N_r U \cap N_r V) \longrightarrow \varinjlim_r \underset{\sim}{K}(R; N_r U) \vee \varinjlim_r \underset{\sim}{K}(R; N_r V) \\ \longrightarrow \underset{\sim}{K}(R; X), \end{aligned}$$

when $U \cup V = X$. Using this Mayer-Vietoris sequence, one can now proceed as follows to produce a target for a map out from $\underset{\sim}{K}(R; X)$. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ be any finite covering of X. We define a simplicial complex $\beta(\mathcal{U})$ by letting the vertices of $\beta(\mathcal{U})$ be in 1-1 correspondence with A, so $V(\beta(\mathcal{U})) = \{v_\alpha\}_{\alpha \in A}$, and declaring that $\{v_{\alpha_1}, v_{\alpha_2}, \ldots, v_{\alpha_s}\}$ spans a simplex if and only if there is a number r so that the subset $N_r U_{\alpha_1} \cap N_r U_{\alpha_2} \cap \cdots \cap N_r U_{\alpha_s} \subseteq X$ is unbounded. The existence of the above Mayer-Vietoris sequence permits the construction of a map $\underset{\sim}{K}(R; X) \xrightarrow{\tau(\mathcal{U})} \Sigma\beta(\mathcal{U}) \wedge \underset{\sim}{K}(R)$. After some technical work, one obtains a map $\tau: \underset{\sim}{K}(R; X) \to \operatorname{holim}_{\mathcal{U}} \Sigma\beta(\mathcal{U}) \wedge \underset{\sim}{K}(R)$, the homotopy inverse limit over a category of coverings. In fact, this category may be taken to be directed, and the resulting prospectrum is often rather easy to understand.

One can now use slight elaborations of the above ideas to prove that if $\Gamma = \pi_1 X$, where X is closed, compact, and admits a metric of non-positive curvature, then $\alpha: B\Gamma_+ \wedge \underset{\sim}{K}\mathbf{Z} \to \underset{\sim}{K}\mathbf{Z}\Gamma$ is injective onto a summand. The elaboration consists of building the correct Γ-equivariant version of the map τ. The important fact derived from the curvature condition is that two distinct geodesics emanating from the same point grow infinitely far apart. In outline, the proof goes as follows. One constructs a so-called locally-finite homology spectrum of $\widetilde{X}$ with coefficients in $\underset{\sim}{K}\mathbf{Z}$, $h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z})$, which has the following properties.

(a) $h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z})^\Gamma \cong \underset{\sim}{K}\mathbf{Z} \wedge B\Gamma_+$;

(b) $h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z})^{h\Gamma} \cong h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z})^\Gamma$;

(c) $h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z}) \cong S^n \wedge \underset{\sim}{K}\mathbf{Z}$, where $n = \dim X$;

(d) there exists a map $\overline{\alpha}: h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z}) \to \underset{\sim}{K}(\mathbf{Z}; \widetilde{X})$, so that the induced map on fixed points is the assembly map α;

(e) the map $h_{\mathrm{lf}}(\widetilde{X}; \underset{\sim}{K}\mathbf{Z}) \to \underset{\sim}{K}(\mathbf{Z}, \widetilde{X})$ is onto a Γ-equivariantly split summand.

It is in proving part (e) that one uses the map τ above. The curvature condition allows one to choose coverings judiciously, so as to prove (e).

3 Concluding Remarks

(i) One can attack the question of whether α is in fact an equivalence by these methods as well. One must show that the map $\underset{\sim}{K}(\mathbf{Z}\Gamma) \to \underset{\sim}{K}(\mathbf{Z};\widetilde{X})^{h\Gamma}$ is also split injective. This one can do by studying the bounded K-theory of the universal covering space of the stable normal bundle to X. One must also compute precisely $\underset{\sim}{K}(\mathbf{Z},\widetilde{X})$. The precise results one obtains will appear in due course [4].

(ii) The condition that the manifold be closed should be removable in many cases. In particular, I expect that results will be obtained for arithmetic groups.

(iii) The method is not restricted to fundamental groups of manifolds. In particular, the so-called Bruhat-Tits buildings are metric spaces which are not manifolds. The computation of their bounded K-theory should give results on cocompact discrete subgroups over p-adic fields, and S-arithmetic groups.

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