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On L^1-approximation

CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York Port Chester

Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1989

Printed in Great Britain at the University Press, Cambridge

British Cataloguing in Publication Data

Pinkus, Allan, 1946–
On L^1–approximation.
1. Mathematics. Functional analysis.
Approximation
I. Title
515.7

Library of Congress Cataloguing in publication data

Pinkus, Allan, 1946–
On L^1–approximation.
(Cambridge tracts in mathematics; 93)
On t.p. 1 is superscript.
Bibliography: p.
Includes indexes.
1. Approximation theory. 2. Least absolute
deviations (Statistics) I. Title. II. Series.
QA211.P55 1989 511'.4 88-35343
ISBN 0-521-36650-X

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1
Preliminaries

In this chapter we introduce certain basic general facts from approximation theory. These will be used in later chapters. The topics we touch upon are the classic problems of approximation theory, namely existence, characterization, uniqueness, and continuity of the best approximation operator. We assume that most of these results are familiar to the reader, for they are contained, in one form or another, in various introductory texts on approximation theory. For the sake of completeness, at the very least, we also include most of their proofs. Most readers should skim the contents of this chapter simply to familiarize themselves with the notation and certain definitions.

1. Existence

We first fix some notation. X will always denote a normed linear space over the reals $\mathbb{R}$. A subset Y of X is given. Our problem is to approximate elements $f \in X$ from elements of Y. The 'error' in this problem we denote by

$$E(f;Y) = \inf\{\|f-g\| : g \in Y\}.$$

The subset Y of X is said to be an *existence set* for X (often termed a *proximinal set*) if to each $f \in X$ there exists a $g^* \in Y$ for which

$$\|f-g^*\| \le \|f-g\|$$

for all $g \in Y$, i.e., for each $f \in X$ the above infimum is attained. Such g^* (if they exist) are called *best approximants* to f from Y.

Much is known concerning existence sets. However we shall only review some very elementary results.

Theorem 1.1. *Let Y be a compact subset of X. Then Y is an existence set for X.*

Proof. Let $f \in X$ and

$$E = E(f;Y) = \inf\{\|f-g\| : g \in Y\}.$$

From the definition of E, there exists a sequence $\{g_n\}$, $g_n \in Y$, with the property that $\lim_{n\to\infty} \|f-g_n\| = E$.

Since Y is compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ which converges to a $g^* \in Y$, i.e., $\lim_{k\to\infty} \|g_{n_k} - g^*\| = 0$. Now, for every k,

$$\|f-g^*\| \le \|f-g_{n_k}\| + \|g_{n_k}-g^*\|.$$

The left-hand-side of the inequality is independent of k. Let $k \to \infty$. The first term on the right-hand-side approaches E, while the second term tends to zero. Thus

$$\|f - g^*\| \leq \lim_{k\to\infty} [\|f - g_{n_k}\| + \|g_{n_k} - g^*\|] = E.$$

However $\|f - g^*\| \geq E$ since $g^* \in Y$. Hence $\|f - g^*\| = E$, and g^* is a best approximant to f from Y. □

It is not necessary that Y be compact in order for it to be an existence set.

Theorem 1.2. *Let C be a closed subset of a finite-dimensional subspace U of X. Then C is an existence set for X.*

Proof. Let $f \in X$ and $v \in C$. When best approximating f from C, it suffices to consider only those $u \in C$ for which

$$\|f - u\| \leq \|f - v\| = M.$$

Define

$$A = \{u : u \in C, \|f - u\| \leq M\}.$$

Let $\|f\| = N$. Then, for each $u \in A$,

$$\|u\| \leq \|f\| + \|f - u\| \leq N + M.$$

Thus A is a closed, bounded subset of U. Any closed, bounded subset of a finite-dimensional subspace is compact. From Theorem 1.1, A is an existence set for X. We therefore have a $u^* \in A$ for which

$$\|f - u^*\| \leq \|f - u\|, \quad \text{all } u \in A.$$

This in turn implies that

$$\|f - u^*\| \leq \|f - u\|, \quad \text{all } u \in C.$$ □

As a special case of the above theorem we have a classic result which, for convenience, we now formally state.

Corollary 1.3. *Let U be a finite-dimensional subspace of a normed linear space X. Then U is an existence set for X.*

2. Characterization

We present here two types of characterization theorems. The first is based on the one-sided Gateaux derivatives, while the second is a consequence of the Hahn-Banach Theorem.

Let $f, g \in X$. If

$$\lim_{t\to 0} \frac{\|f + tg\| - \|f\|}{t}$$

exists, then the limit is said to be the Gateaux derivative of f in the direction g. Such limits do not necessarily exist. However the one-sided limits always exist.

Proposition 1.4. *Let $f, g \in X$, and set*

$$r(t) = \frac{\|f + tg\| - \|f\|}{t}.$$

On $(0, \infty), r(t)$ is a non-decreasing function of t and is bounded below.

Proof. We first prove that $r(t)$ is bounded below on $(0, \infty)$. From the triangle inequality,

$$\|f + tg\| \geq \|f\| - \|tg\| = \|f\| - t\|g\|.$$

Thus for $t \in (0, \infty), r(t) \geq -\|g\|$.

It remains to prove that $r(t)$ is non-decreasing on $(0, \infty)$. Let $0 < s < t < \infty$. Then,

$$t\|f + sg\| = \|tf + stg\| = \|s(f + tg) + (t - s)f\| \leq s\|f + tg\| + (t - s)\|f\|.$$

Thus

$$t(\|f + sg\| - \|f\|) \leq s(\|f + tg\| - \|f\|),$$

whence we obtain $r(s) \leq r(t)$. □

For $f, g \in X$, set

$$\tau_+(f, g) = \lim_{t \to 0^+} \frac{\|f + tg\| - \|f\|}{t}.$$

On the basis of Proposition 1.4, $\tau_+(f, g)$ exists for every $f, g \in X$.

Our first characterization theorem now follows.

Theorem 1.5. *Let M be a linear subspace of X, and $f \in X \backslash \overline{M}$. Then $g^* \in M$ is a best approximant to f from M if and only if $\tau_+(f - g^*, g) \geq 0$ for all $g \in M$.*

Proof. ($\Rightarrow$). Assume g^* is a best approximant to f from M. Since M is a subspace,

$$\|f - g^* + tg\| \geq \|f - g^*\|$$

for every $g \in M$ and $t \in \mathbb{R}$. Thus $\tau_+(f - g^*, g) \geq 0$, essentially by definition.
($\Leftarrow$). Assume $\tau_+(f - g^*, g) \geq 0$ for all $g \in M$. From Proposition 1.4, $r(t)$ is a non-decreasing function of t on $(0, \infty)$. Setting $t = 1$ and remembering that M is a linear subspace, we obtain

$$\|f - g\| - \|f - g^*\| \geq \tau_+(f - g^*, g^* - g) \geq 0.$$

Thus $\|f - g\| \geq \|f - g^*\|$ for all $g \in M$. □

A totally analogous proof allows us to obtain this next result.

Theorem 1.6. *Let K be a convex subset of X, and assume $f \in X \backslash \overline{K}$. Then g^* is a best approximant to f from K if and only if $\tau_+(f - g^*, g^* - g) \geq 0$ for all $g \in K$.*

Heuristically the above technique should be considered as a generalized perturbation technique. The next set of results, although formally equivalent, are more in the spirit of separating hyperplanes.

For a normed linear space X over $\mathbb{R}$, let X^* denote the space of continuous (bounded) real-valued linear functionals on X. For $h \in X^*$,

$$|||h||| = \sup\{|h(f)| : f \in X, \|f\| \leq 1\}$$

defines a norm on X^*. With this norm, X^* is a Banach space, i.e., is complete.

The Hahn-Banach (Extension) Theorem, in one of its simpler forms, may be stated as follows.

Theorem 1.7 (Hahn-Banach). *Let M be a linear subspace of X. Assume H is a continuous linear functional on M. There then exists an $h \in X^*$ for which*

$$1)\ h(g) = H(g), \ \text{all } g \in M$$
$$2)\ |||h|||_X = |||H|||_M.$$

The subscript on the $|||\cdot|||$ in (2) indicates where this norm is taken. In general it is well understood from the context and is deleted. One consequence of the above result is:

Proposition 1.8. *Let M be a linear subspace of X. Assume $f \in X$, and*

$$E(f; M)\ (= E) = \inf\{\|f - g\| : g \in M\} > 0.$$

There exists an $h \in X^*$ satisfying

$$1)\ h(g) = 0, \ \text{all } g \in M$$
$$2)\ |||h||| = 1$$
$$3)\ h(f) = E.$$

Proof. Let L denote the linear span of M and f. Define a continuous linear functional H on L as follows: For $\alpha \in \mathbb{R}$, $g \in M$,

$$H(\alpha f + g) = \alpha E.$$

By definition $H(g) = 0$ for all $g \in M$, and $H(f) = E$. Furthermore, it is easily seen that $|||H|||_L = 1$. Now apply Theorem 1.7. □

As a consequence of Proposition 1.8 we have this next main result paralleling Theorem 1.5.

Theorem 1.9. *Let M be a linear subspace of X, and $f \in X\backslash\overline{M}$. Then g^* is a best approximant to f from M if and only if there exists an $h \in X^*$ for which*

$$1)\ h(g) = 0, \ \text{all } g \in M$$

$$2)\ |||h||| = 1$$

$$3)\ h(f - g^*) = \|f - g^*\|.$$

Proof. Since $f \in X\backslash\overline{M}$,

$$E = \inf\{\|f - g\| : g \in M\} > 0.$$

($\Rightarrow$). Assume g^* is a best approximant to f from M. Thus $\|f - g^*\| = E$. Let h be as given by Proposition 1.8. Then (1) and (2) are valid. Furthermore, from (1) and (3) of Proposition 1.8,

$$\|f - g^*\| = E = h(f) = h(f - g^*).$$

Thus (3) of this theorem holds.
($\Leftarrow$). Assume that (1), (2) and (3) hold. Let $g \in M$. Then,

$$\|f - g^*\| = h(f - g^*) = h(f - g) \leq |||h||| \cdot \|f - g\| = \|f - g\|.$$

Thus g^* is a best approximant to f from M. □

Remark. Note that $\widetilde{g}$ is any other best approximant to f from M if and only if $h(f - \widetilde{g}) = \|f - \widetilde{g}\|$ for the h satisfying (1), (2) and (3) of Theorem 1.9.

To obtain a result parallel to Theorem 1.6, we use the following generalization of the Hahn-Banach Theorem.

Theorem 1.10 (Basic Separation Theorem). *Let A and B be disjoint convex subsets of X. Assume A has interior. There exists a non-zero $h \in X^*$ and $c \in \mathbb{R}$ such that $h(f) \geq c$ for all $f \in A$, and $h(f) \leq c$ for all $f \in B$.*

This next result also generalizes Theorem 1.9.

Theorem 1.11. *Let K be a convex subset of X, and assume $f \in X\backslash\overline{K}$. Then g^* is a best approximant to f from K if and only if there exists an $h \in X^*$ satisfying*

$$1)\ h(g^*) \geq h(g), \ \text{all } g \in K$$

$$2)\ |||h||| = 1$$

$$3)\ h(f - g^*) = \|f - g^*\|.$$

Proof. ($\Rightarrow$). Assume

$$\|f - g^*\| = \inf\{\|f - g\| : g \in K\} = E > 0.$$

Let $A = \{f_0 : f_0 \in X, \|f - f_0\| < E\}$. The sets A and K satisfy the conditions of Theorem 1.10. As such there exists an $\widetilde{h} \in X^*, \widetilde{h} \neq 0$, and a $\widetilde{c} \in \mathbb{R}$ for which $\widetilde{h}(f_0) \geq \widetilde{c}$ for all $f_0 \in A$ and $\widetilde{h}(g) \leq \widetilde{c}$ for all $g \in K$. By continuity, $\widetilde{h}(f_0) \geq \widetilde{c}$ for all $f_0 \in \overline{A}$, and therefore $\widetilde{h}(g^*) = \widetilde{c}$.

Translating by $\widetilde{c}$, there exists a $c \in \mathbb{R}$ ($c = \widetilde{h}(f) - \widetilde{c}$) with the property that $\widetilde{h}(f - f_0) \leq c$ for all $f_0 \in \overline{A}$, and $\widetilde{h}(f - g) \geq c$ for all $g \in K$. Since A is a ball of positive radius about f, we necessarily have $c > 0$. Set $h = (E/c)\widetilde{h}$. It is now easily checked that h satisfies (1), (2) and (3).
($\Leftarrow$). Assume (1), (2) and (3) hold for some $h \in X^*$ and $g^* \in K$. Then, for any $g \in K$,

$$\|f - g^*\| = h(f - g^*) \leq h(f - g) \leq |||h||| \cdot \|f - g\| = \|f - g\|,$$

and g^* is a best approximant to f from K. □

If K is a convex cone, i.e., $g \in K$ implies $\alpha g \in K$ for all $\alpha > 0$, then Theorem 1.11 can be somewhat sharpened.

Corollary 1.12. *Let K be a convex cone in X. Assume that $f \in X \backslash \overline{K}$. Then g^* is a best approximant to f from K if and only if there exists an $h \in X^*$ satisfying*

$$1)\ 0 = h(g^*) \geq h(g), \ \text{all } g \in K$$
$$2)\ |||h||| = 1$$
$$3)\ h(f - g^*) = \|f - g^*\|,$$

or, equivalently,

$$1')\ 0 \geq h(g), \ \text{all } g \in K$$
$$2')\ |||h||| = 1$$
$$3')\ h(f) = \|f - g^*\|.$$

3. Uniqueness and Strong Uniqueness

Let Y be a subset of X. For each $f \in X$, set

$$P_Y(f) = \{g^* : g^* \in Y, \|f - g^*\| = E(f; Y)\}.$$

$P_Y(f)$ is the subset of Y containing all the best approximants to f from Y. $P_Y(f)$ may, of course, be the empty set. $P_Y(f)$ is not the empty set for every $f \in X$ if and only if Y is an existence set for X. In general P_Y is a set-valued map from X onto Y. P_Y is referred to as the *metric projection* onto Y. We are naturally interested in the various general properties enjoyed by P_Y. These depend on both Y and X. Convexity is one geometric property which the metric projection inherits directly from Y.

Proposition 1.13. *If K is convex, then $P_K(f)$ is convex for each $f \in X$.*

Proof. Assume $f \in X$ and $P_K(f)$ contains at least two distinct elements g_1 and g_2. Then $E(f; K) = E = \|f - g_i\|$, $i = 1, 2$. For each $\lambda \in [0, 1]$, set $g_\lambda = \lambda g_1 + (1 - \lambda) g_2$. Then

$$\|f - g_\lambda\| \leq \lambda \|f - g_1\| + (1 - \lambda) \|f - g_2\| = E.$$

Since K is convex, $g_\lambda \in K$, and $\|f - g_\lambda\| \geq E$. Thus $\|f - g_\lambda\| = E$ and, by definition, $g_\lambda \in P_K(f)$. □

We continue to assume that K is a convex subset of X. For each $f \in X$, $P_K(f)$ may be the empty set, a unique element of K, or a convex subset of K containing more than one element. In certain cases, simple norm properties eliminate this third option, i.e., $P_K(f)$ will contain at most one element.

Definition 1.1. The normed linear space X is said to be *strictly convex* if, for any $f, g \in X$ satisfying $f \neq g$ and $\|f\| = \|g\| = 1$, we have $\|\lambda f + (1-\lambda)g\| < 1$ for every $\lambda \in (0, 1)$.

The above definition is equivalent to the statement that if $\|f\| = \|g\| = \|(f+g)/2\|$, then $f = g$.

Theorem 1.14. *Assume K is a convex subset of a strictly convex normed linear space X. Then, for each $f \in X$, $P_K(f)$ contains at most one element.*

Proof. Assume that $g_1, g_2 \in P_K(f)$, and $E = \|f - g_i\|$, $i = 1, 2$. From the proof of Proposition 1.13, it follows that $\|\lambda(f - g_1) + (1 - \lambda)(f - g_2)\| = \lambda\|f - g_1\| + (1 - \lambda)\|f - g_2\| = E$ for all $\lambda \in [0, 1]$. Since the norm is strictly convex, this implies that $f - g_1 = f - g_2$, i.e., $g_1 = g_2$. □

Strict convexity is a global property of the norm. A local property dependent on the one-sided Gateaux derivatives will sometimes give even more.

Assume that for a given $f \in X$ there exists a best approximant g^* from Y. Thus

$$\|f - g^*\| \leq \|f - g\|$$

for all $g \in Y$. If, in addition, there exists a $\gamma > 0$ for which

$$\gamma\|g - g^*\| \leq \|f - g\| - \|f - g^*\|$$

for all $g \in Y$, then we say that g^* is a *strongly unique* best approximant to f from Y. The reason for this terminology is simply that 'strong uniqueness' is stronger than 'uniqueness'. If g^* is a strongly unique best approximant to f from Y, then it is most certainly the unique best approximant to f from Y. The converse need not and generally does not hold. If strong uniqueness is present, then we shall denote by $\gamma(f)$ the largest constant satisfying the above inequality.

Strong uniqueness and the identification of $\gamma(f)$ is intimately connected with one-sided Gateaux derivatives. We state and prove our result for a subspace M of X.

Theorem 1.15. *Let M be a subspace of X and $f \in X \backslash \overline{M}$. Assume g^* is a best approximant to f from M. Set*

$$\gamma = \inf\{\tau_+(f - g^*, g) \,:\, g \in M\,,\ \|g\| = 1\}\,.$$

Then $\gamma \geq 0$ *and, for all* $g \in M$,

$$\gamma\|g - g^*\| \leq \|f - g\| - \|f - g^*\|\,.$$

Furthermore, if $\gamma' > \gamma$ *there exists a* $\widetilde{g} \in M$ *for which*

$$\gamma'\|\widetilde{g} - g^*\| > \|f - \widetilde{g}\| - \|f - g^*\|\,.$$

Therefore strong uniqueness holds if and only if $\gamma > 0$, *and in this case* $\gamma(f) = \gamma$.

Proof. Since g^* is a best approximant to f from M, we have from Theorem 1.5 that $\gamma \geq 0$. Assume $\gamma > 0$. From Exercise 2(a) and the definition of γ, we have

$$\tau_+(f - g^*, -g) \geq \gamma\|g\|$$

for all $g \in M$. From Proposition 1.4,

$$\frac{\|f - g^* - tg\| - \|f - g^*\|}{t} \geq \gamma\|g\|$$

for all $t > 0$ and $g \in M$. Therefore

$$\|f - g^* - g\| - \|f - g^*\| \geq \gamma\|g\|$$

for all $g \in M$. Since M is a subspace, we immediately obtain

$$\gamma\|g - g^*\| \leq \|f - g\| - \|f - g^*\|$$

for all $g \in M$.

Assume $\gamma' > \gamma$. By definition there exists a $\overline{g} \in M$, $\|\overline{g}\| = 1$, for which

$$\tau_+(f - g^*, -\overline{g}) < \gamma'\|\overline{g}\|\,.$$

Thus for $t_0 > 0$, sufficiently small,

$$\|f - g^* - t_0\overline{g}\| - \|f - g^*\| < \gamma'\|t_0\overline{g}\|\,.$$

Set $\widetilde{g} = g^* + t_0\overline{g}$. Then

$$\gamma'\|\widetilde{g} - g^*\| > \|f - \widetilde{g}\| - \|f - g^*\|\,,$$

which proves the theorem. □

4. Continuity

Let Y be a subset of X, and recall that

$$E(f;Y) = \inf\{\|f - g\| : g \in Y\}.$$

The first simple fact to be shown is the following.

Proposition 1.16. *$E(f;Y)$ is a continuous function of f. In fact, for any $f_1, f_2 \in X$,*

$$|E(f_1;Y) - E(f_2;Y)| \le \|f_1 - f_2\|.$$

Proof. Assume without loss of generality that $E(f_1;Y) \ge E(f_2;Y)$. Given $\varepsilon > 0$, let $g_0 \in Y$ satisfy

$$\|f_2 - g_0\| \le E(f_2;Y) + \varepsilon.$$

Such a g_0 necessarily exists. Then

$$E(f_1;Y) \le \|f_1 - g_0\| \le \|f_1 - f_2\| + \|f_2 - g_0\| \le \|f_1 - f_2\| + E(f_2;Y) + \varepsilon.$$

Thus, for every $\varepsilon > 0$, we have

$$E(f_1;Y) - E(f_2;Y) \le \|f_1 - f_2\| + \varepsilon$$

which implies the desired result. □

An important application of Proposition 1.16 is in this next result.

Theorem 1.17. *Assume Y is a subset of X. Let $f, f_n \in X$, $n \in \mathbb{N}$, satisfy $\lim_{n\to\infty} \|f - f_n\| = 0$. Assume $g_n \in P_Y(f_n)$ all n, and there exists a $g \in Y$ for which $\lim_{n\to\infty} \|g - g_n\| = 0$. Then $g \in P_Y(f)$.*

Proof. From the triangle inequality,

$$\|f - g\| \le \|f - f_n\| + \|f_n - g_n\| + \|g_n - g\|$$

for all n. By assumption, $\lim_{n\to\infty} \|f - f_n\| = \lim_{n\to\infty} \|g - g_n\| = 0$. From Proposition 1.16,

$$\lim_{n\to\infty} \|f_n - g_n\| = \lim_{n\to\infty} E(f_n;Y) = E(f;Y).$$

Thus $\|f - g\| \le E(f;Y)$. Since $g \in Y$, this implies that $g \in P_Y(f)$. □

From Theorem 1.17 we deduce the following.

Proposition 1.18. *Let U be a finite-dimensional subspace of X. Assume $f, f_n \in X$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} \|f - f_n\| = 0$. Further assume that $P_U(f) = \{u^*\}$. Then, for any choice of $u_n \in P_U(f_n)$, we have $\lim_{n\to\infty} \|u_n - u^*\| = 0$.*

Proof. Since $\lim_{n\to\infty} \|f - f_n\| = 0$, there exists a $c \in \mathbb{R}$ such that $\|f_n\| \le c$ for all n. Thus $\|u_n\| \le 2c$ for all n. Each element of the sequence $\{u_n\}$ is in the compact set

$$U \cap \{g : g \in X, \|g\| \le 2c\}.$$

Thus there exists a subsequence of $\{u_n\}$ which converges to some $u \in U$. From Theorem 1.17, $u = u^*$. Since this is valid for any convergent subsequence, it follows that $\lim_{n\to\infty} \|u_n - u^*\| = 0$. □

Before ending this chapter we make the following formal definition.

Definition 1.2. Let Y be an existence set for X. Then Y is said to be a *unicity set* if $P_Y(f)$ is a singleton for all $f \in X$. That is, to each $f \in X$ there exists a unique best approximant from Y. If Y is a subspace of X and a unicity set, then we shall say that Y is a *unicity space.*

Some authors use the term unicity set, without the assumption of Y being an existence set, to mean that $P_Y(f)$ contains at most one element for each $f \in X$. Other authors use the terms Chebyshev and semi-Chebyshev, respectively. In this work the term Chebyshev will be used in a different context.

As an immediate application of Proposition 1.18, we have:

Corollary 1.19. *Let U be a finite-dimensional unicity space of X. Then the single-valued operator $P_U(\cdot)$ is continuous on X.*

That is, if $f, f_n \in X$, and $\lim_{n\to\infty} \|f - f_n\| = 0$, then necessarily $\lim_{n\to\infty} \|P_U(f) - P_U(f_n)\| = 0$.

Exercises

1. Prove that, if Y is an existence set for X, then Y is closed.

2. Prove that, for every $f, g, h \in X$,

$$a)\ \tau_+(f, \alpha g) = \alpha\, \tau_+(f, g) \text{ for all } \alpha \geq 0;$$
$$b)\ \tau_+(f, g + h) \leq \tau_+(f, g) + \tau_+(f, h).$$

3. For $f, g \in X$, set

$$\tau_-(f, g) = \lim_{t\to 0^-} \frac{\|f + tg\| - \|f\|}{t}.$$

Prove that $\tau_-(\cdot\,,\cdot)$ always exists and $\tau_-(f, g) = -\tau_+(f, -g)$.

4. Let $f \in X$, $f \neq 0$. Assume that

$$\tau(f, g) = \lim_{t\to 0} \frac{\|f + tg\| - \|f\|}{t}$$

exists for all $g \in X$. Prove that $\tau(f, \cdot) \in X^*$, i.e., $\tau(f, \cdot)$ is a continuous linear functional on X.

5. Assume that $\tau(f, g)$ exists for all $f, g \in X$, with $f \neq 0$. Let M be a linear subspace of X and $f \in X \backslash \overline{M}$. Prove that $g^* \in P_M(f)$ if and only if $\tau(f - g^*, g) = 0$ for all $g \in M$.

6. Let X be an inner product space and $f \neq 0$. Prove that

$$\tau(f,g) = (f,g)/\|f\|.$$

7. Prove Theorem 1.6.

8. Let M be a linear subspace of X. Assume $f \in X$, and

$$E = \inf\{\|f-g\| \,:\, g \in M\} > 0\,.$$

Prove that

$$E = \max\{h(f) \,:\, h \in X^*,\ |||h||| \leq 1,\ h(g) = 0,\ \text{all } g \in M\}.$$

9. Prove Corollary 1.12.

10. Let K be a convex subset of X and $f \in X\backslash\overline{K}$. Prove that g^* is a strongly unique best approximant to f from K if and only if

$$\inf\{\tau_+(f-g^*, g^*-g)/\|g^*-g\| \,:\, g \in K\,, g \neq g^*\} > 0\,.$$

11. Let U be a finite-dimensional subspace of X. Prove that, for each $f \in X$, $P_U(f)$ is bounded and closed (hence compact).

12. Let $u_1, \ldots, u_n$ be a basis for the n-dimensional subspace U of X. Set

$$H(a_1, \ldots, a_n) = \left\|f - \sum_{i=1}^{n} a_i u_i\right\|.$$

Prove that H is continuous, convex, and $\lim_{|||\mathbf{a}|||\to\infty} H(\mathbf{a}) = \infty$, where $|||\cdot|||$ is any norm on $\mathbb{R}^n$ and $\mathbf{a} = (a_1, \ldots, a_n)$.

13. For $H(\mathbf{a})$ as in Exercise 12, set

$$A = \{\mathbf{a}^* \,:\, H(\mathbf{a}^*) = \min_{\mathbf{a}} H(\mathbf{a})\}.$$

Prove that A is a convex, closed, bounded subset of $\mathbb{R}^n$.

Notes and References

Most of the material of this chapter may be found in either Sections 1 of Chapter I and Appendix I of Singer [1970], Chapter 1 of Cheney [1966], or Chapter 1 of Watson [1980]. Additional material on Gateaux derivatives is contained in Dunford, Schwartz [1958, pp.445–451] and Chapter 26 of Köthe [1969]. There is a direct interconnection between linear functionals and what we have called one-sided Gateaux derivatives. It is given by the fact that the range of $h(g)$ for $h \in X^*$ satisfying $|||h||| = 1$ and $h(f) = \|f\|$, is exactly the interval $[-\tau_+(f,-g), \tau_+(f,g)]$ (see Dunford, Schwartz [1958, p.447] and Köthe [1969, p.349]). The Basic Separation Theorem was lifted from Dunford, Schwartz [1958, p.417]. Singer [1970] is the best reference for a historical development of this material. The concept of strong uniqueness was introduced by Newman, Shapiro [1963]. The approach taken here may be found in Papini [1978], see also Wulbert [1971].

If the functional $\tau(f,g)$ of Exercise 4 exists for all $f,g \in X$, $f \neq 0$, then the space X is said to be *smooth.* This corresponds to the existence of a unique $h \in X^*$ satisfying $|||h||| = 1$ and $h(f) = \|f\|$ for each $f \in X$, $f \neq 0$. From Exercise 5 we have that in a smooth space strong uniqueness never holds with respect to any subspace. Smoothness and strict convexity are essentially dual concepts (Köthe [1969, p.346]). If X^* is strictly convex (smooth), then X is smooth (strictly convex). If X is a reflexive Banach space, then the converse holds. This is one explanation for the fact that in any L^p space, $1 < p < \infty$, strong uniqueness from a subspace never holds.