Second Order Partial Differential Equations in Hilbert Spaces

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Contents

Preface x

I THEORY IN SPACES OF CONTINUOUS FUNCTIONS 1

1 Gaussian measures 3
  1.1 Introduction and preliminaries .......................... 3
  1.2 Definition and first properties of Gaussian measures .... 7
    1.2.1 Measures in metric spaces .......................... 7
    1.2.2 Gaussian measures .................................. 8
    1.2.3 Computation of some Gaussian integrals .......... 11
    1.2.4 The reproducing kernel ............................ 12
  1.3 Absolute continuity of Gaussian measures ................. 17
    1.3.1 Equivalence of product measures in $\mathbb{R}^\infty$ ... 18
    1.3.2 The Cameron-Martin formula ........................ 22
    1.3.3 The Feldman-Hajek theorem ........................ 24
  1.4 Brownian motion ...................................... 27

2 Spaces of continuous functions 30
  2.1 Preliminary results ................................... 30
  2.2 Approximation of continuous functions .................. 33
  2.3 Interpolation spaces .................................. 36
    2.3.1 Interpolation between $UC_0(H)$ and $UC_0^1(H)$ ..... 36
    2.3.2 Interpolatory estimates ............................ 39
    2.3.3 Additional interpolation results .................... 42

3 The heat equation 44
  3.1 Preliminaries ........................................ 44
  3.2 Strict solutions ..................................... 48
### Contents

3.3 Regularity of generalized solutions ........................................ 54
  3.3.1 $Q$-derivatives .............................................. 54
  3.3.2 $Q$-derivatives of generalized solutions .......................... 57
3.4 Comments on the Gross Laplacian ......................................... 67
3.5 The heat semigroup and its generator .................................... 69

4 Poisson’s equation ..................................................................... 76
  4.1 Existence and uniqueness results ........................................... 76
  4.2 Regularity of solutions ..................................................... 78
  4.3 The equation $\Delta_Q u = g$ ............................................. 83
    4.3.1 The Liouville theorem ............................................ 87

5 Elliptic equations with variable coefficients ............................... 90
  5.1 Small perturbations ....................................................... 90
  5.2 Large perturbations ...................................................... 93

6 Ornstein-Uhlenbeck equations .................................................. 99
  6.1 Existence and uniqueness of strict solutions .......................... 100
  6.2 Classical solutions ...................................................... 103
  6.3 The Ornstein-Uhlenbeck semigroup ..................................... 111
    6.3.1 $\pi$-Convergence ............................................... 112
    6.3.2 Properties of the $\pi$-semigroup ($R_t$) ....................... 113
    6.3.3 The infinitesimal generator ..................................... 114
  6.4 Elliptic equations ...................................................... 116
    6.4.1 Schauder estimates ............................................... 119
    6.4.2 The Liouville theorem .......................................... 121
  6.5 Perturbation results for parabolic equations ........................ 122
  6.6 Perturbation results for elliptic equations .......................... 124

7 General parabolic equations ................................................... 127
  7.1 Implicit function theorems ............................................. 128
  7.2 Wiener processes and stochastic equations .......................... 131
    7.2.1 Infinite dimensional Wiener processes ....................... 131
    7.2.2 Stochastic integration .......................................... 132
  7.3 Dependence of the solutions to stochastic equations on initial data ......................................................... 133
    7.3.1 Convolution and evaluation maps ................................ 133
    7.3.2 Solutions of stochastic equations ............................ 138
  7.4 Space and time regularity of the generalized solutions .......... 139
  7.5 Existence ............................................................... 142
Contents

7.6 Uniqueness .................................................. 144
  7.6.1 Uniqueness for the heat equation ..................... 145
  7.6.2 Uniqueness in the general case ...................... 146
7.7 Strong Feller property .................................... 150

8 Parabolic equations in open sets 156
  8.1 Introduction .............................................. 156
  8.2 Regularity of the generalized solution ................. 158
  8.3 Existence theorems ...................................... 165
  8.4 Uniqueness of the solutions ........................... 178

II THEORY IN SOBOLEV SPACES 185

9 $L^2$ and Sobolev spaces 187
  9.1 Itô-Wiener decomposition ............................... 188
    9.1.1 Real Hermite polynomials ......................... 188
    9.1.2 Chaos expansions ................................. 190
    9.1.3 The space $L^2(H, \mu; H)$ ....................... 193
  9.2 Sobolev spaces ......................................... 194
    9.2.1 The space $W^{1,2}(H, \mu)$ ...................... 196
    9.2.2 Some additional summability results .............. 197
    9.2.3 Compactness of the embedding $W^{1,2}(H, \mu) \subset L^2(H, \mu)$ 198
    9.2.4 The space $W^{2,2}(H, \mu)$ ..................... 201
  9.3 The Malliavin derivative ................................ 203

10 Ornstein-Uhlenbeck semigroups on $L^p(H, \mu)$ 205
  10.1 Extension of $(R_t)$ to $L^p(H, \mu)$ ................ 206
    10.1.1 The adjoint of $(R_t)$ in $L^2(H, \mu)$ ........ 211
  10.2 The infinitesimal generator of $(R_t)$ ............... 212
    10.2.1 Characterization of the domain of $L_2$ ....... 215
  10.3 The case when $(R_t)$ is strong Feller ............... 217
    10.3.1 Additional regularity properties of $(R_t)$ ..... 221
    10.3.2 Hypercontractivity of $(R_t)$ .................. 224
  10.4 A representation formula for $(R_t)$ in terms of the second quantization operator ....................... 228
    10.4.1 The second quantization operator ............... 228
    10.4.2 The adjoint of $(R_t)$ ......................... 230
  10.5 Poincaré and log-Sobolev inequalities .................. 230
    10.5.1 The case when $M = 1$ and $Q = I$ ............. 232
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.5.2</td>
<td>A generalization</td>
<td>235</td>
</tr>
<tr>
<td>10.6</td>
<td>Some additional regularity results when $Q$ and $A$ commute</td>
<td>236</td>
</tr>
<tr>
<td>11</td>
<td>Perturbations of Ornstein-Uhlenbeck semigroups</td>
<td>238</td>
</tr>
<tr>
<td>11.1</td>
<td>Bounded perturbations</td>
<td>239</td>
</tr>
<tr>
<td>11.2</td>
<td>Lipschitz perturbations</td>
<td>245</td>
</tr>
<tr>
<td>11.2.1</td>
<td>Some additional results on the Ornstein-Uhlenbeck semigroup</td>
<td>251</td>
</tr>
<tr>
<td>11.2.2</td>
<td>The semigroup $(P_t)$ in $L^p(H,\nu)$</td>
<td>256</td>
</tr>
<tr>
<td>11.2.3</td>
<td>The integration by parts formula</td>
<td>260</td>
</tr>
<tr>
<td>11.2.4</td>
<td>Existence of a density</td>
<td>263</td>
</tr>
<tr>
<td>12</td>
<td>Gradient systems</td>
<td>267</td>
</tr>
<tr>
<td>12.1</td>
<td>General results</td>
<td>268</td>
</tr>
<tr>
<td>12.1.1</td>
<td>Assumptions and setting of the problem</td>
<td>268</td>
</tr>
<tr>
<td>12.1.2</td>
<td>The Sobolev space $W^{1,2}(H,\nu)$</td>
<td>271</td>
</tr>
<tr>
<td>12.1.3</td>
<td>Symmetry of the operator $N_0$</td>
<td>272</td>
</tr>
<tr>
<td>12.1.4</td>
<td>The $m$-dissipativity of $N_1$ on $L^1(H,\nu)$</td>
<td>274</td>
</tr>
<tr>
<td>12.2</td>
<td>The $m$-dissipativity of $N_2$ on $L^2(H,\nu)$</td>
<td>277</td>
</tr>
<tr>
<td>12.3</td>
<td>The case when $U$ is convex</td>
<td>281</td>
</tr>
<tr>
<td>12.3.1</td>
<td>Poincaré and log-Sobolev inequalities</td>
<td>288</td>
</tr>
<tr>
<td>13</td>
<td>Applications to Control Theory</td>
<td>291</td>
</tr>
<tr>
<td>13</td>
<td>Second order Hamilton-Jacobi equations</td>
<td>293</td>
</tr>
<tr>
<td>13.1</td>
<td>Assumptions and setting of the problem</td>
<td>296</td>
</tr>
<tr>
<td>13.2</td>
<td>Hamilton-Jacobi equations with a Lipschitz Hamiltonian</td>
<td>300</td>
</tr>
<tr>
<td>13.2.1</td>
<td>Stationary Hamilton-Jacobi equations</td>
<td>302</td>
</tr>
<tr>
<td>13.3</td>
<td>Hamilton-Jacobi equation with a quadratic Hamiltonian</td>
<td>305</td>
</tr>
<tr>
<td>13.3.1</td>
<td>Stationary equation</td>
<td>308</td>
</tr>
<tr>
<td>13.4</td>
<td>Solution of the control problem</td>
<td>310</td>
</tr>
<tr>
<td>13.4.1</td>
<td>Finite horizon</td>
<td>310</td>
</tr>
<tr>
<td>13.4.2</td>
<td>Infinite horizon</td>
<td>312</td>
</tr>
<tr>
<td>13.4.3</td>
<td>The limit as $\varepsilon \to 0$</td>
<td>314</td>
</tr>
<tr>
<td>14</td>
<td>Hamilton-Jacobi inclusions</td>
<td>316</td>
</tr>
<tr>
<td>14.1</td>
<td>Introduction</td>
<td>316</td>
</tr>
<tr>
<td>14.2</td>
<td>Excessive weights and an existence result</td>
<td>317</td>
</tr>
<tr>
<td>14.3</td>
<td>Weak solutions as value functions</td>
<td>324</td>
</tr>
</tbody>
</table>
Contents

14.4 Excessive measures for Wiener processes ................. 328

IV APPENDICES .............. 333

A Interpolation spaces .......... 335
  A.1 The interpolation theorem .......... 335
  A.2 Interpolation between a Banach space X and the domain of
      a linear operator in X ............... 336

B Null controllability .......... 338
  B.1 Definition of null controllability .......... 338
  B.2 Main results ...................... 339
  B.3 Minimal energy ..................... 340

C Semiconcave functions and Hamilton-Jacobi semigroups .......... 347
  C.1 Continuity modulus .................... 347
  C.2 Semiconcave and semiconvex functions .......... 348
  C.3 The Hamilton-Jacobi semigroups ............... 351

Bibliography .......... 358

Index .......... 376
Chapter 1

Gaussian measures

This chapter is devoted to some basic results on Gaussian measures on separable Hilbert spaces, including the Cameron-Martin and Feldman-Hajek formulae. The greater part of the results are presented with complete proofs.

1.1 Introduction and preliminaries

We are given a real separable Hilbert space $H$ (with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$). The space of all linear bounded operators from $H$ into $H$, equipped with the operator norm $\|\cdot\|$, will be denoted by $L(H)$. If $T \in L(H)$, then $T^*$ is the adjoint of $T$. Moreover, by $L^+(H)$ we shall denote the subset of $L(H)$ consisting of all nonnegative symmetric operators. Finally, we shall denote by $\mathcal{B}(H)$ the $\sigma$-algebra of all Borel subsets of $H$.

Before introducing Gaussian measures we need some results about trace class and Hilbert-Schmidt operators.

A linear bounded operator $R \in L(H)$ is said to be of trace class if there exist two sequences $(a_k), (b_k)$ in $H$ such that

$$Ry = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \quad y \in H, \quad (1.1.1)$$

and

$$\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty. \quad (1.1.2)$$

Notice that if (1.1.2) holds then the series in (1.1.1) is norm convergent. Moreover, it is not difficult to show that $R$ is compact.
We shall denote by $L_1(H)$ the set of all operators of $L(H)$ of trace class. $L_1(H)$, endowed with the usual linear operations, is a Banach space with the norm

$$
\|R\|_{L_1(H)} = \inf \left\{ \sum_{k=1}^{\infty} |a_k| |b_k| : R y = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k, \ y \in H, \ (a_k), (b_k) \subset H \right\}.
$$

We set $L_1^+(H) = L^+(H) \cap L_1(H)$. If an operator $R$ is of trace class then its trace, $\text{Tr} \ R$, is defined by the formula

$$
\text{Tr} \ R = \sum_{j=1}^{\infty} \langle Re_j, e_j \rangle,
$$

where $(e_j)$ is an orthonormal and complete basis on $H$. Notice that, if $R$ is given by (1.1.1), we have

$$
\text{Tr} \ R = \sum_{j=1}^{\infty} \langle a_j, b_j \rangle.
$$

Thus the definition of the trace is independent on the choice of the basis and

$$
|\text{Tr} \ R| \leq \|R\|_{L_1(H)}.
$$

**Proposition 1.1.1** Let $S \in L_1(H)$ and $T \in L(H)$. Then

(i) $ST, TS \in L_1(H)$ and

$$
\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|, \ \|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|.
$$

(ii) $\text{Tr}(ST) = \text{Tr}(TS)$.

**Proof.** (i) Assume that $Sy = \sum_{k=1}^{\infty} \langle y, a_k \rangle b_k$, $y \in H$, where $\sum_{k=1}^{\infty} |a_k| |b_k| < +\infty$. Then

$$
STy = \sum_{k=1}^{\infty} \langle y, T^* a_k \rangle b_k, \ y \in H,
$$

and

$$
\sum_{k=1}^{\infty} |T^* a_k| |b_k| \leq \|T\| \sum_{k=1}^{\infty} |a_k| |b_k|.
$$
Gaussian measures

It is therefore clear that $ST \in L_1(H)$ and $\|ST\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|$. Similarly we can prove that $\|TS\|_{L_1(H)} \leq \|S\|_{L_1(H)} \|T\|$.

(ii) From part (i) it follows that

$$\text{Tr}(ST) = \sum_{k=1}^{\infty} \langle b_k, T^* a_k \rangle = \sum_{k=1}^{\infty} \langle Tb_k, a_k \rangle.$$ 

In the same way $\text{Tr}(TS) = \sum_{k=1}^{\infty} \langle a_k, Tb_k \rangle$, and the conclusion follows. \(\square\)

We say that $R \in L(H)$ is of Hilbert-Schmidt class if there exists an orthonormal and complete basis $(e_k)$ in $H$ such that

$$\sum_{k,j=1}^{\infty} |\langle Se_k, e_j \rangle|^2 < +\infty. \quad (1.1.3)$$

If (1.1.3) holds then we have

$$\sum_{k=1}^{\infty} |Se_k|^2 = \sum_{k,j=1}^{\infty} |\langle Se_k, e_j \rangle|^2 = \sum_{k,j=1}^{\infty} |\langle e_k, S^* e_j \rangle|^2 = \sum_{j=1}^{\infty} |S^* e_j|^2. \quad (1.1.4)$$

Now if $(f_k)$ is another complete orthonormal basis in $H$, we have

$$\sum_{m=1}^{\infty} |Sf_m|^2 = \sum_{m,n=1}^{\infty} |\langle Sf_m, e_n \rangle|^2 = \sum_{m,n=1}^{\infty} |\langle f_m, S^* e_n \rangle|^2 = \sum_{n=1}^{\infty} |S^* e_n|^2.$$ 

Thus, by (1.1.4) we see that the assertion (1.1.3) is independent of the choice of the complete orthonormal basis $(e_k)$. We shall denote by $L_2(H)$ the space of all Hilbert-Schmidt operators on $H$. $L_2(H)$, endowed with the norm

$$\|S\|_{L_2(H)}^2 = \sum_{k,j=1}^{\infty} |\langle Se_k, e_j \rangle|^2 = \sum_{k=1}^{\infty} |Se_k|^2,$$

is a Banach space.

**Proposition 1.1.2** Let $S, T \in L_2(H)$. Then $ST \in L_1(H)$ and

$$\|ST\|_{L_1(H)} \leq \|S\|_{L_2(H)} \|T\|_{L_2(H)}. \quad (1.1.5)$$
Proof. Let $(e_k)$ be a complete and orthonormal basis in $H$, then

$$Ty = \sum_{k=1}^{\infty} \langle Ty, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle e_k,$$

$$STy = \sum_{k=1}^{\infty} \langle y, T^* e_k \rangle Se_k.$$

Consequently $ST \in L_1(H)$ and

$$\|ST\|_{L_1(H)} \leq \sum_{k=1}^{\infty} |T^* e_k| |Se_k| \leq \left( \sum_{k=1}^{\infty} |T^* e_k|^2 \right)^{1/2} \left( \sum_{k=1}^{\infty} |Se_k|^2 \right)^{1/2} = \|T\|_{L_2(H)} \|S\|_{L_2(H)}.$$

Therefore the conclusion follows. □

**Warning.** If $S$ and $T$ are bounded operators, and $ST$ is of trace class then in general $TS$ is not, as the following example, provided by S. Peszat [183], shows.

Define two linear operators $S$ and $T$ on the product space $H \times H$, by

$$S = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \quad T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$ST = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad TS = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

and it is enough to take $B$ of trace class and $A$ not of trace class. □

We have also the following result, see e.g. A. Pietsch [187].

**Proposition 1.1.3** Assume that $S$ is a compact self-adjoint operator, and that $(\lambda_k)$ are its eigenvalues (repeated according to their multiplicity).

(i) $S \in L_1(H)$ if and only if $\sum_{k=1}^{\infty} |\lambda_k| < +\infty$. Moreover $\|S\|_{L_1(H)} = \sum_{k=1}^{\infty} |\lambda_k|$, and $\text{Tr } S = \sum_{k=1}^{\infty} \lambda_k$.

(ii) $S \in L_2(H)$ if and only if $\sum_{k=1}^{\infty} |\lambda_k|^2 < +\infty$. Moreover

$$\|S\|_{L_2(H)} = \left( \sum_{k=1}^{\infty} |\lambda_k|^2 \right)^{1/2}.$$
More generally let $S$ be a compact operator on $H$. Denote by $(\lambda_k)$ the sequence of all positive eigenvalues of the operator $(S^*S)^{1/2}$, repeated according to their multiplicity. Denote by $L_p(H), \ p > 0$, the set of all operators $S$ such that

$$\|S\|_{L_p(H)} = \left( \sum_{k=1}^{\infty} \lambda_k^p \right)^{1/p} < +\infty. \quad (1.1.6)$$

Operators belonging to $L_1(H)$ and $L_2(H)$ are precisely the trace class and the Hilbert-Schmidt operators.

The following result holds, see N. Dunford and J. T. Schwartz [107].

**Proposition 1.1.4** Let $S \in L_p(H), \ T \in L_q(H)$ with $p > 0, q > 0$. Then $ST \in L_r(H)$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and

$$\|TS\|_{L_r(H)} \leq 2^{1/r} \|S\|_{L_p(H)} \|T\|_{L_q(H)}. \quad (1.1.7)$$

### 1.2 Definition and first properties of Gaussian measures

#### 1.2.1 Measures in metric spaces

If $E$ is a metric space, then $\mathcal{B}(E)$ will denote the Borel $\sigma$-algebra, that is the smallest $\sigma$-algebra of subsets of $E$ which contains all closed (open) subsets of $E$.

Let metric spaces $E_1, E_2$ be equipped with $\sigma$-fields $\mathcal{E}_1, \mathcal{E}_2$ respectively. Measurable mappings $X : E_1 \rightarrow E_2$ will often be called random variables. If $\mu$ is a measure on $(E_1, \mathcal{E}_1)$, then its image by the transformation $X$ will be denoted by $X \circ \mu$:

$$X \circ \mu(A) = \mu(X^{-1}(A)), \ A \in \mathcal{E}_2.$$ 

We call $X \circ \mu$ the law or the distribution of $X$, and we set $X \circ \mu = \mathcal{L}(X)$.

If $\nu$ and $\mu$ are two finite measures on $(E, \mathcal{E})$ such that $\Gamma \in \mathcal{E}$, $\mu(\Gamma) = 0$ implies $\nu(\Gamma) = 0$ then one writes $\nu << \mu$ and one says that $\nu$ is absolutely continuous with respect to $\mu$. If there exist $A, B \in \mathcal{E}$ such that $A \cap B = \emptyset$, $\mu(A) = \nu(B) = 1$, one says that $\mu$ and $\nu$ are singular.

If $\nu << \mu$ then by the Radon-Nikodým theorem there exists $g \in L^1(E, \mathcal{E}, \mu)$ nonnegative such that

$$\nu(\Gamma) = \int_{\Gamma} g(x) \mu(dx), \ \Gamma \in \mathcal{E}.$$
The function $g$ is denoted by $\frac{d\nu}{d\mu}$.

If $\nu \ll \mu$ and $\mu \ll \nu$ then one says that $\mu$ and $\nu$ are equivalent and writes $\mu \sim \nu$.

We have the following change of variable formula. If $\varphi$ is a nonnegative measurable real function on $E_2$, then

$$
\int_{E_1} \varphi(X(x))\mu(dx) = \int_{E_2} \varphi(y)X \circ \mu(dy). \quad (1.2.1)
$$

Let $\mu$ and $\nu$ be two measures on a separable Hilbert space $H$; if $T \circ \mu = T \circ \nu$ for any linear operator $T : H \to \mathbb{R}^n$, $n \in \mathbb{N}$, then $\mu = \nu$.

Random variables $X_1, \ldots, X_n$ are said to be independent if

$$
\mathcal{L}(X_1, \ldots, X_n) = \mathcal{L}(X_1) \times \cdots \times \mathcal{L}(X_n).
$$

A family of random variables $(X_\alpha)_{\alpha \in A}$ is said to be independent, if any finite subset of the family is independent.

Probability measures on a separable Hilbert space $H$ will always be regarded as defined on $\mathcal{B}(H)$. If $\mu$ is a probability measure on $H$, then its Fourier transform is defined by

$$
\hat{\mu}(\lambda) = \int_H e^{i\langle \lambda, x \rangle} \mu(dx), \ \lambda \in H;
$$

$\hat{\mu}$ is called the characteristic function of $\mu$. One can show that if the characteristic functions of two measures are identical, then the measures are identical as well.

### 1.2.2 Gaussian measures

We first define Gaussian measures on $\mathbb{R}$. If $a \in \mathbb{R}$ we set

$$
N_{a,0}(dx) = \delta_a(dx),
$$

where $\delta_a$ is the Dirac measure at $a$. If moreover $\lambda > 0$ we set

$$
N_{a,\lambda}(dx) = \frac{1}{\sqrt{2\pi \lambda}} e^{-\frac{(x-a)^2}{2\lambda}} dx.
$$

The Fourier transform of $N_{a,\lambda}$ is given by

$$
\mathcal{N}_{a,\lambda}(h) = \int_{\mathbb{R}} e^{ihx} N_{a,\lambda}(dx) = e^{iah - \frac{1}{2} \lambda h^2}, \ h \in \mathbb{R}.
$$
More generally we show now that in an arbitrary separable Hilbert space and for arbitrary \( Q \in L^+_1(H) \) there exists a unique measure \( N_{a,Q} \) such that

\[
\hat{N}_{a,Q}(h) = \int_H e^{i(h,x)} N_{a,Q}(dx) = e^{i(h,a)} e^{-\frac{1}{2} \langle Qh,h \rangle}, \ h \in H.
\]

Let in fact \( Q \in L^+_1(H) \). Then there exist a complete orthonormal system \( (e_k) \) on \( H \) and a sequence of nonnegative numbers \( (\lambda_k) \) such that \( Qe_k = \lambda_k e_k, \ k \in \mathbb{N} \). We set \( x_h = \langle x, e_h \rangle, \ h \in \mathbb{N} \), and \( P_n x = \sum_{k=1}^n x_k e_k, \ x \in H, \ n \in \mathbb{N} \). Let us introduce an isomorphism \( \gamma \) from \( H \) into \( \ell^2 \): \( (1) \)

\[
x \in H \rightarrow \gamma(x) = (x_k) \in \ell^2.
\]

In the following we shall always identify \( H \) with \( \ell^2 \). In particular we shall write \( P_n x = (x_1, ..., x_n), \ x \in \ell^2 \).

A subset \( I \) of \( H \) of the form \( I = \{ x \in H : (x_1, ..., x_n) \in B \} \), where \( B \in \mathcal{B}(\mathbb{R}^n) \), is said to be cylindrical. It is easy to see that the \( \sigma \)-algebra generated by all cylindrical subsets of \( H \) coincides with \( \mathcal{B}(H) \).

**Theorem 1.2.1** Let \( a \in H, \ Q \in L^+_1(H) \). Then there exists a unique probability measure \( \mu \) on \( (H, \mathcal{B}(H)) \) such that

\[
\int_H e^{i(h,x)} \mu(dx) = e^{i(h,a)} e^{-\frac{1}{2} \langle Qh,h \rangle}, \ h \in H. \tag{1.2.2}
\]

Moreover \( \mu \) is the restriction to \( H \) (identified with \( \ell^2 \)) of the product measure

\[
\times_{k=1}^\infty \mu_k = \times_{k=1}^\infty N_{a_k,\lambda_k},
\]

defined on \( (\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)) \). \( (2) \)

We set \( \mu = N_{a,Q} \), and call \( a \) the **mean** and \( Q \) the **covariance operator** of \( \mu \). Moreover \( N_{0,Q} \) will be denoted by \( N_Q \).

**Proof of Theorem 1.2.1.** Since a characteristic function uniquely determines the measure, we have only to prove existence.

Let us consider the sequence of Gaussian measures \( (\mu_k) \) on \( \mathbb{R} \) defined as

\[
\mu_k = N_{a_k,\lambda_k}, \ k \in \mathbb{N}, \text{ and the product measure } \mu = \times_{k=1}^\infty \mu_k \text{ in } \mathbb{R}^\infty,
\]

---

1. For any \( p \geq 1 \), we denote by \( \ell^p \) the Banach space of all sequences \( (x_k) \) of real numbers such that \( |x|_p := (\sum_{k=1}^\infty |x_k|^p)^{1/p} < +\infty \).
2. We shall consider \( \mathbb{R}^\infty \) as a metric space with the distance \( d(x,y) := \sum_{k=1}^\infty 2^{-k} \frac{|x_k-y_k|}{1+|x_k-y_k|}, \ x, y \in \mathbb{R}^\infty \).
We want to prove that \( \mu \) is concentrated on \( \ell^2 \), (that it is clearly a Borel subset of \( \mathbb{R}^\infty \)). For this it is enough to show that

\[
\int_{\ell^\infty} |x|^2 \mu(dx) < +\infty.
\]

We have in fact, by the monotone convergence theorem,

\[
\int_{\ell^\infty} |x|^2 \mu(dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} x_k^2 \mu(dx) = \sum_{k=1}^{\infty} \left( \int_{\mathbb{R}} (x_k - a_k)^2 \mu_k(dx) + a_k^2 \right)
\]

\[
= \sum_{k=1}^{\infty} (\lambda_k + a_k^2) = \text{Tr} \; Q + |a|^2 < +\infty.
\]

Now we consider the restriction of \( \mu \) to \( \ell^2 \), which we still denote by \( \mu \). We have to prove that (1.2.2) holds. Setting \( \nu_n = \prod_{k=1}^{n} \mu_k \), we have

\[
\int_{\ell^2} e^{i \langle x, h \rangle} \mu(dx) = \lim_{n \to \infty} \int_{\ell^2} e^{i \langle P_n h, P_n x \rangle} \mu(dx)
\]

\[
= \lim_{n \to \infty} \int_{\mathbb{R}^n} e^{i \langle P_n h, P_n x \rangle} \nu_n(dx) = \lim_{n \to \infty} e^{i \langle P_n h, P_n a \rangle - \frac{1}{2} \langle Q P_n h, P_n h \rangle}
\]

\[
= e^{i \langle h, a \rangle - \frac{1}{2} \langle Q h, h \rangle}.
\]

If the law of a random variable is a Gaussian measure, then the random variable is called Gaussian. It easily follows from Theorem 1.2.1 that a random variable \( X \) with values in \( H \) is Gaussian if and only if for any \( h \in H \) the real valued random variable \( \langle h, X \rangle \) is Gaussian.

**Remark 1.2.2** From the proof of Theorem 1.2.1 it follows that

\[
\int_{H} |x|^2 N_{a,Q}(dx) = \text{Tr} \; Q + |a|^2.
\]

**Proposition 1.2.3** Let \( T \in L(H) \), and \( a \in H \), and let \( \Gamma x = T x + a, \; x \in H \). Then \( \Gamma \circ N_{m,Q} = N_{T m + a, T Q T^*} \).

**Proof.** Notice that, by the change of variables formula (1.2.1), we have

\[
\int_{H} e^{i \langle \lambda, y \rangle} \Gamma \circ N_{m,Q}(dy) = \int_{H} e^{i \langle \lambda, \Gamma x \rangle} N_{m,Q}(dy)
\]

\[
= \int_{H} e^{i \langle \lambda, T x + a \rangle} N_{m,Q}(dy) = e^{i \langle \lambda, a \rangle} e^{i \langle T^* \lambda, m \rangle - \frac{1}{2} \langle Q T^* \lambda, T^* \lambda \rangle}.
\]

This shows the result. \( \square \)
1.2.3 Computation of some Gaussian integrals

We are here given a Gaussian measure $N_{a,Q}$. We set

$$L^2(H, N_{a,Q}) = L^2(H, B(H), N_{a,Q}).$$

The following identities can be easily proved, using (1.2.2).

**Proposition 1.2.4** We have

$$\int H x N_{a,Q}(dx) = a, \quad (1.2.5)$$

$$\int H \langle x - a, y \rangle \langle x - a, z \rangle N_{a,Q}(dx) = \langle Qy, z \rangle. \quad (1.2.6)$$

$$\int H |x - a|^2 N_{a,Q}(dx) = \text{Tr } Q. \quad (1.2.7)$$

**Proof.** We prove as instance (1.2.6). We have

$$\int_H x N_{a,Q}(dx) = \lim_{n \to \infty} \int_H P_n x N_{a,Q}(dx).$$

But

$$\int_H P_n x N_{a,Q}(dx) = (2\pi)^{-n/2} \prod_{k=1}^{n} \int_{\mathbb{R}} x_k \lambda_k^{-1/2} e^{-\frac{(x_k-a_k)^2}{2\lambda_k}} dx_k = a_k,$$

and the conclusion follows. $\square$

**Proposition 1.2.5** For any $h \in H$, the exponential function $E_h$, defined as

$$E_h(x) = e^{(h,x)}, \quad x \in H,$$

belongs to $L^p(H, N_{a,Q})$, $p \geq 1$, and

$$\int_H e^{(h,x)} N_{a,Q}(dx) = e^{(a,h)} e^{\frac{1}{2} \langle Qh, h \rangle}. \quad (1.2.8)$$

Moreover the subspace of $L^2(H, N_{a,Q})$ spanned by all $E_h$, $h \in H$, is dense on $L^2(H, N_{a,Q})$.

**Proof.** We have

$$\int_H e^{(P_n h, P_n x)} N_{a,Q}(dx) = e^{(P_n a, P_n h)} e^{\frac{1}{2} \langle Q P_n h, P_n h \rangle}.$$
Chapter 1

Letting \( n \) tend to 0 this gives (1.2.8).

Let us prove the last statement. Let \( \varphi \in L^2(H, N_{a,Q}) \) be such that

\[
\int_H e^{(h,x)} \varphi(x) N_{a,Q}(dx) = 0, \quad h \in H.
\]

Denote by \( \varphi^+ \) and \( \varphi^- \) the positive and negative parts of \( \varphi \). Then

\[
\int_H e^{(h,x)} \varphi^+(x) N_{a,Q}(dx) = \int_H e^{(h,x)} \varphi^-(x) N_{a,Q}(dx), \quad h \in H.
\]

Let us define two measures

\[
\mu(dx) = \varphi^+(x) N_{a,Q}(dx), \quad \nu(dx) = \varphi^-(x) N_{a,Q}(dx).
\]

Then \( \mu \) and \( \nu \) are finite measures such that

\[
\int_H e^{(h,x)} \mu(dx) = \int_H e^{(h,x)} \nu(dx), \quad h \in H.
\]

Let \( T \) be any linear transformation from \( H \) into \( \mathbb{R}^n, n \in \mathbb{N} \). Then for any \( \lambda \in \mathbb{R}^n \)

\[
\int_{\mathbb{R}^n} e^{(\lambda,z)} T \circ \mu(dz) = \int_H e^{(\lambda,Tx)} \mu(dx) = \int_H e^{(T^*\lambda,z)} \mu(dx)
\]

\[
= \int_H e^{(T^*\lambda,x)} \nu(dx) = \int_{\mathbb{R}^n} e^{(\lambda,z)} T \circ \nu(dz).
\]

By a well known finite dimensional result \( T \circ \mu = T \circ \nu \). Consequently measures \( \mu \) and \( \nu \) are identical and so \( \varphi = 0 \). \( \Box \)

1.2.4 The reproducing kernel

Here we are given an operator \( Q \in L^1_1(H) \). We denote as before by \( (e_k) \) a complete orthonormal system in \( H \) and by \( (\lambda_k) \) a sequence of positive numbers such that \( Qe_k = \lambda_k e_k, \ k \in \mathbb{N} \).

The subspace \( Q^{1/2}(H) \) is called the reproducing kernel of the measure \( N_Q \). If \( \text{Ker} \ Q = \{0\} \), \( Q^{1/2}(H) \) is dense on \( H \). In fact, if \( x_0 \in H \) is such that \( (Q^{1/2}h,x_0) = 0 \) for all \( h \in H \), we have \( Q^{1/2}x_0 = 0 \) and so \( Qx_0 = 0 \), which yields \( x_0 = 0 \).

Let \( \text{Ker} \ Q = \{0\} \). We are now going to introduce an isomorphism \( W \) from \( H \) into \( L^2(H, N_Q) \) that will play an important rôle in the following. The isomorphism \( W \) is defined by

\[
f \in Q^{1/2}(H) \rightarrow Wf \in L^2(H, N_Q), \quad Wf(x) = (Q^{-1/2}f, x), \ x \in H.
\]
Gaussian measures

By (1.2.7) it follows that
\[
\int_H W_f(x) W_g(x) N_Q(dx) = \langle f, g \rangle, \ f, g \in H.
\]
Thus \( W \) is an isometry and it can be uniquely extended to all of \( H \). It will be denoted by the same symbol. For any \( f \in H \), \( W_f \) is a real Gaussian random variable \( N_{|f|^2} \).

More generally, for arbitrary elements \( f_1, \ldots, f_n \), \((W_{f_1}, \ldots, W_{f_n})\) is a Gaussian vector with mean 0 and covariance matrix \( \langle f_i, f_j \rangle \). If \( \text{Ker } Q \neq \{0\} \) then the transformation \( f \to W_f \) can be defined in exactly the same way but only for \( f \in H_0 = Q^{1/2}(H) \). We will write in some cases \( \langle Q^{-1/2} y, f \rangle \) instead of \( W_f(y) \).

The proof of the following proposition is left as an exercise to the reader.

**Proposition 1.2.6** For any orthonormal sequence \((f_n)\) in \( H \), the family
\[
1, W_{f_n}, W_{f_k} W_{f_l}, 2^{-1/2} (W_{f_m}^2 - 1), \ m, n, k, l \in \mathbb{N}, \ k \neq l,
\]
is orthonormal in \( L^2(H, N_Q) \).

Next we consider the function \( f \to e^{W_f} \).

**Proposition 1.2.7** The transformation \( f \to e^{W_f} \) acts continuously from \( H \) into \( L^2(H, N_Q) \), and
\[
\int_H e^{W_f(x)} N_Q(dx) = e^{\frac{1}{2} |f|^2},
\]
(1.2.9)
\[
\int_H e^{\lambda W_f(x)} N_Q(dx) = e^{-\frac{1}{2} \lambda^2 |f|^2}, \ \lambda \in \mathbb{R}.
\]

**Proof.** Since \( W_f \) is Gaussian with law \( N_{|f|^2} \), (1.2.9) follows. Moreover, taking into account (1.2.8) it follows that
\[
\int_H \left( e^{W_f} - e^{W_g} \right)^2 dN_Q = \int_H \left[ e^{2W_f} - 2 e^{W_f+g} + e^{2W_g} \right] dN_Q
\]
\[
eq e^{2|f|^2} - 2 e^{\frac{1}{2} |f+g|^2} + e^{2|g|^2} = \left[ e^{|f|^2} - e^{|g|^2} \right]^2 + 2 e^{|f|^2+|g|^2} \left[ 1 - e^{-\frac{1}{2} |f-g|^2} \right],
\]
which shows that \( W_f \) is locally uniformly continuous on \( H \). \( \square \)

Let us define the determinant of \( 1 + S \) where \( S \) is a compact self-adjoint operator in \( L_1(H) \):
\[
\det (1 + S) = \prod_{k=1}^{\infty} (1 + s_k),
\]
where \((s_k)\) is the sequence of eigenvalues of \(S\) (repeated according to their multiplicity).

**Proposition 1.2.8** Assume that \(M\) is a symmetric operator such that \(Q^{1/2}MQ^{1/2} < 1\), \((3)\) and let \(b \in H\). Then

\[
\int_H \exp \left\{ \frac{1}{2} \langle My, y \rangle + \langle b, y \rangle \right\} N_Q(dy)
= \left[ \det(1 - Q^{1/2}MQ^{1/2}) \right]^{-1/2} \exp \left\{ \frac{1}{2} |(1 - Q^{1/2}MQ^{1/2})^{-1/2}Q^{1/2}b|^2 \right\}.
\]

**Proof.** Let \((g_n)\) be an orthonormal basis for the operator \(Q^{1/2}MQ^{1/2}\), and let \((\gamma_n)\) be the sequence of the corresponding eigenvalues.

**Claim 1.** We have

\[
\langle b, x \rangle = \sum_{k=1}^{\infty} \langle Q^{1/2}b, g_n \rangle W_{g_n}(x), \text{ } N_Q\text{-a.e.}
\]

**Claim 2.** We have

\[
\langle Mx, x \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{g_n}(x)|^2, \text{ } N_Q\text{-a.e},
\]

the series being convergent in \(L^1(H, N_Q)\).

We shall only prove the more difficult second claim.

Let \(P_N = \sum_{k=1}^{N} e_k \otimes e_k\). \((4)\) Then for any \(x \in H\) we have

\[
\langle MP_Nx, P_Nx \rangle = \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, Q^{-1/2}P_Nx \rangle
= \sum_{n=1}^{\infty} \langle (Q^{1/2}MQ^{1/2})Q^{-1/2}P_Nx, g_n \rangle \langle Q^{-1/2}P_Nx, g_n \rangle
= \sum_{n=1}^{\infty} \gamma_n |\langle Q^{-1/2}P_Nx, g_n \rangle|^2.
\]

Consequently, for each fixed \(x\)

\[
\langle MP_Nx, P_Nx \rangle = \sum_{n=1}^{\infty} \gamma_n |W_{P_Nx}g_n|^2, \text{ } N \in \mathbb{N}.
\]

\(^3\)This means that \((Q^{1/2}MQ^{1/2}x, x) < |x|^2\) for any \(x \in H\) different from 0.

\(^4\)We rember that \((e_k)\) is the sequence of eigenvectors of \(Q\).
Moreover for each $L \in \mathbb{N}$

$$
\int_{H} \left| \langle M_{P_{N}x}, P_{N}x \rangle - \sum_{n=1}^{L} \gamma_{n} |W_{P_{N}g_{n}}|^{2} \right| N_{Q}(dx)
\leq \sum_{n=L+1}^{\infty} |\gamma_{n}| \int_{H} |W_{P_{N}g_{n}}|^{2} N_{Q}(dx)
= \sum_{n=L+1}^{\infty} |\gamma_{n}| |P_{N}g_{n}|^{2} \leq \sum_{n=L+1}^{\infty} |\gamma_{n}|.
$$

As $N \to \infty$ then $P_{N}x \to x$ and $W_{P_{N}g_{n}} \to W_{g_{n}}$ in $L^{2}(H, N_{Q})$. Passing to subsequences if needed, and using the Fatou lemma, we see that

$$
\int_{H} \left| \langle M_{x}, x \rangle - \sum_{n=1}^{L} \gamma_{n} |W_{g_{n}}(x)|^{2} \right| N_{Q}(dx) \leq \sum_{n=L+1}^{\infty} |\gamma_{n}|.
$$

Therefore the claim is proved.

By the claims it follows that

$$
\exp \left\{ \frac{1}{2} \langle M_{x}, x \rangle + \langle b, x \rangle \right\}
= \lim_{L \to \infty} \exp \left\{ \sum_{n=1}^{L} \frac{1}{2} \gamma_{n} |W_{g_{n}}(x)|^{2} + \langle Q^{1/2}b, g_{n} \rangle W_{g_{n}}(x) \right\},
$$

with a.e. convergence with respect to $N_{Q}$ for a suitable subsequence. Using the fact that $(W_{g_{n}})$ are independent Gaussian random variables, we obtain, by a direct calculation, for $p \geq 1$,

$$
\int_{H} \exp \left\{ p \sum_{n=1}^{L} \frac{1}{2} \gamma_{n} |W_{g_{n}}(x)|^{2} + p(Q^{1/2}b, g_{n}) W_{g_{n}}(x) \right\} N_{Q}(dx)
= \left[ \prod_{n=1}^{L} (1 - p\gamma_{n}) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_{n} \rangle|^{2}}{1 - p\gamma_{n}} \right\}.
$$
Since $\gamma_n < 1$, and $\sum_{n=1}^{\infty} |\gamma_n| < \infty$, there exists $p > 1$ such that $p\gamma_n < 1$, for all $n \in \mathbb{N}$. Therefore

$$
\lim_{L \to \infty} \prod_{n=1}^{L} (1 - p\gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\}
= \left[ \prod_{n=1}^{\infty} (1 - p\gamma_n) \right]^{-1/2} \exp \left\{ \frac{1}{2} \sum_{n=1}^{\infty} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - p\gamma_n} \right\}.
$$

So the sequence $\left( \exp \left\{ \sum_{n=1}^{L} \left[ \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} \right)$ is uniformly integrable. Consequently, passing to the limit, we find

$$
\int_{H} \exp \left\{ 1/2 \langle My, y \rangle + \langle b, y \rangle \right\} N_Q(dy)
= \lim_{L \to \infty} \int_{H} \exp \left\{ \sum_{n=1}^{L} \left[ \frac{1}{2} \gamma_n |W_{g_n}(x)|^2 + \langle Q^{1/2}b, g_n \rangle W_{g_n}(x) \right] \right\} N_Q(dx)
= \lim_{L \to \infty} \prod_{n=1}^{L} (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\}
= \prod_{n=1}^{\infty} (1 - \gamma_n)^{-1/2} \exp \left\{ \frac{1}{2} \frac{|\langle Q^{1/2}b, g_n \rangle|^2}{1 - \gamma_n} \right\}
= \left( \det (1 - Q^{1/2}MQ^{1/2}) \right)^{-1/2} \exp \left\{ \frac{1}{2} (1 - Q^{1/2}MQ^{1/2})^{-1/2} |Q^{1/2}b|^2 \right\}.
$$

**Remark 1.2.9** It follows from the proof of the proposition that

$$
\langle Mx, x \rangle = \sum_{k=1}^{\infty} \gamma_n W_{g_n}^2(x) = \sqrt{2} \sum_{k=1}^{\infty} \gamma_n \left[ 2^{-1/2}(W_{g_n}^2(x) - 1) \right] + \sum_{k=1}^{\infty} \gamma_n,
$$
and so, by Proposition 1.2.6, we have

\[
\int_H [\langle Mx, x \rangle]^2 N_Q(dx) = 2 \sum_{k=1}^{\infty} \gamma_n^2 + \left( \sum_{k=1}^{\infty} \gamma_n \right)^2
\]

\[
= 2\|Q^{1/2}MQ^{1/2}\|_{L_2(H)}^2 + (\text{Tr} Q^{1/2}MQ^{1/2})^2
\]

\[
< +\infty.
\]

**Proposition 1.2.10** Let \( T \in L_1(H) \). Then there exists the limit

\[
\langle TQ^{-1/2}y, Q^{-1/2}y \rangle := \lim_{n \to \infty} \langle TQ^{-1/2}P_ny, Q^{-1/2}P_ny \rangle, \quad N_Q-a.e.,
\]

where \( P_n = \sum_{k=1}^{n} e_k \otimes e_k \).

Moreover we have the following expansion in \( L^2(H, N_Q) \):

\[
\langle TQ^{-1/2}y, Q^{-1/2}y \rangle = \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle + \sum_{m \neq n=1}^{\infty} \langle Tg_n, g_m \rangle W_{g_n} W_{g_m}
\]

\[
\times \sqrt{2} \sum_{n=1}^{\infty} \langle Tg_n, g_n \rangle \left[ 2^{-1/2} (W_{g_n}^2 - 1) \right]. \quad (1.2.11)
\]

The proof of the following result is similar to that of Claim 2 in the proof of Proposition 1.2.8 and it is left to the reader.

**Proposition 1.2.11** Assume that \( M \) is a symmetric trace-class operator such that \( M < 1 \),\(^{5}\) and \( b \in H \). Then

\[
\int_H \exp \left\{ \frac{1}{2} \langle MQ^{-1/2}y, Q^{-1/2}y \rangle + \langle b, Q^{-1/2}y \rangle \right\} N_Q(dy)
\]

\[
= (\det(1 - M))^{-1/2} e^{\frac{1}{2} \|1-M\|^{-1/2}b^2}. \quad (1.2.12)
\]

### 1.3 Absolute continuity of Gaussian measures

We consider here two Gaussian measures \( \mu, \nu \). We want to prove the Feldman-Hajek theorem, that is they are either singular or equivalent.

\(^{5}\)That is \( \langle Mx, x \rangle < |x|^2 \) for all \( x \neq 0 \).
In §1.3.1 we recall some results on equivalence of measures on $\mathbb{R}^\infty$ including the Kakutani theorem. In §1.3.2 we consider the case when $\mu = N_Q$ and $\nu = N_{a,Q}$ with $Q \in L^+_1(H)$ and $a \in H$, proving the Cameron-Martin formula. Finally in §1.3.3 we consider the more difficult case when $\mu = N_Q$ and $\nu = N_R$ with $Q, R \in L^+_1(H)$.

1.3.1 Equivalence of product measures in $\mathbb{R}^\infty$

It is convenient to introduce the notion of Hellinger integral.

Let $\mu, \nu$ be probability measures on a measurable space $(E, \mathcal{E})$. Then $\lambda = \frac{1}{2}(\mu + \nu)$ is also a probability measure on $(E, \mathcal{E})$ and we have obviously

$$\mu << \lambda, \nu << \lambda.$$ 

We define the Hellinger integral by

$$H(\mu, \nu) = \left( \int_E \left( \frac{d\mu}{d\lambda}(x) \frac{d\nu}{d\lambda}(x) \right) \lambda(dx) \right)^{1/2}.$$ 

Instead of $\frac{1}{2}(\mu + \nu)$ one could choose as $\lambda$ any measure equivalent to $\frac{1}{2}(\mu + \nu)$ without changing the value of $H(\mu, \nu)$.

By using Hölder’s inequality we see that

$$|H(\mu, \nu)|^2 \leq \int_E \frac{d\mu}{d\lambda}(x) \lambda(dx) \int_E \frac{d\nu}{d\lambda}(x) \lambda(dx) = 1,$$

so that $0 \leq H(\mu, \nu) \leq 1$.

**Exercise 1.3.1** (a) Let $\mu = N_q$ and $\nu = N_{a,q}$, where $a \in \mathbb{R}$ and $q > 0$. Show that we have

$$H(\mu, \nu) = e^{-\frac{a^2}{4q}}.$$ 

(1.3.1)

(b) Let $\mu = N_q$ and $\nu = N_\rho$, where $q, \rho > 0$. Show that we have

$$H(\mu, \nu) = \left[ \frac{4q\rho}{(q + \rho)^2} \right]^{1/4}.$$ 

(1.3.2)

**Proposition 1.3.2** Assume that $H(\mu, \nu) = 0$. Then the measures $\mu$ and $\nu$ are singular.
Gaussian measures

**Proof.** Set $\alpha = \frac{d\mu}{d\lambda}$, $\beta = \frac{d\nu}{d\lambda}$. Since $H(\mu, \nu) = \int_{\Omega} \sqrt{\alpha \beta} \ d\lambda = 0$, we have $\alpha \beta = 0$, $\lambda$-a.e. Consequently, setting

$$A = \{ \omega \in \Omega : \alpha(\omega) = 0 \}, \quad B = \{ \omega \in \Omega : \beta(\omega) = 0 \},$$

we have $\lambda(A \cup B) = 1$. This means that $\lambda(C) = 0$ where $C = \Omega \setminus (A \cup B)$, and hence $\mu(C) = \nu(C) = 0$. Then, as

$$\mu(A) = \int_A \alpha \ d\lambda = 0, \quad \nu(B) = \int_B \beta \ d\lambda = 0,$$

we have that $\mu$ and $\nu$ are singular since $\mu(A \cup C) = \nu(B) = 0$, $(A \cup C) \cap B = \emptyset$. $\square$

**Proposition 1.3.3** Let $\mathcal{G} \subset \mathcal{E}$ be a $\sigma$-algebra, and let $\mu_{\mathcal{G}}$ and $\nu_{\mathcal{G}}$ be the restrictions of $\mu$ and $\nu$ to $(\mathcal{E}, \mathcal{G})$. Then we have $H(\mu, \nu) \leq H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}})$.

**Proof.** Let $\lambda_{\mathcal{G}}$ be the restriction of $\lambda$ to $(\mathcal{E}, \mathcal{G})$. It is easy to check that

$$\frac{d\mu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left( \frac{d\mu}{d\lambda} \mid \mathcal{G} \right), \quad \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} = E_{\lambda} \left( \frac{d\nu}{d\lambda} \mid \mathcal{G} \right), \quad \lambda$-a.e.$^6$

Consequently we have (7)

$$H(\mu_{\mathcal{G}}, \nu_{\mathcal{G}}) = \int_E \left[ E_{\lambda} \left( \frac{d\mu}{d\lambda} \mid \mathcal{G} \right) E_{\lambda} \left( \frac{d\nu}{d\lambda} \mid \mathcal{G} \right) \right]^{1/2} d\lambda.$$

Since $\lambda$-a.e.

$$\left[ \frac{d\nu}{d\lambda} \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} \right]^{1/2} \leq \frac{1}{2} \left( \frac{d\nu}{d\lambda} + \frac{d\nu_{\mathcal{G}}}{d\lambda_{\mathcal{G}}} \right),$$

taking conditional expectations of both sides one finds, $\lambda$-a.e.,

$$\left[ E_{\lambda} \left( \frac{d\mu}{d\lambda} \mid \mathcal{G} \right) E_{\lambda} \left( \frac{d\nu}{d\lambda} \mid \mathcal{G} \right) \right]^{1/2} \geq E_{\lambda} \left( \left( \frac{d\mu}{d\lambda} \right)^{1/2} \left( \frac{d\nu}{d\lambda} \right)^{1/2} \mid \mathcal{G} \right). \quad (1.3.3)$$

$^6$ $E_{\lambda}(\eta\mid\mathcal{G})$ is the conditional expectation of the random variable $\eta$ with respect to $\mathcal{G}$ and measure $\lambda$.

$^7$ For positive numbers $a, b, c, d, \sqrt{\frac{ad}{bc}} \leq \frac{1}{2} \left( \frac{a}{b} + \frac{b}{a} \right)$. 
Integrating with respect to $\lambda$ both sides of (1.3.3), the required result follows.

Now let us consider two sequences of measures $(\mu_k)$ and $(\nu_k)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\nu_k \sim \mu_k$ for all $k \in \mathbb{N}$. We set $\lambda_k = \frac{1}{2} (\mu_k + \nu_k)$, and we consider the Hellinger integral

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[ \frac{d\mu_k}{d\lambda_k}(x) \frac{d\nu_k}{d\lambda_k}(x) \right]^{1/2} \lambda_k(dx), \ k \in \mathbb{N}.$$ 

**Remark 1.3.4** Since $(\mu_k)$ and $(\nu_k)$ are equivalent, we have

$$\frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\lambda_k} = \frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\mu_k} \frac{d\mu_k}{d\lambda_k} \frac{d\nu_k}{d\lambda_k} = \frac{d\nu_k}{d\mu_k} \left( \frac{d\mu_k}{d\lambda_k} \right)^2.$$ 

Thus

$$H(\mu_k, \nu_k) = \int_{\mathbb{R}} \left[ \frac{d\nu_k}{d\mu_k}(x) \right]^{1/2} \mu_k(dx). \quad (1.3.4)$$

We also consider the product measures on $\mathbb{R}^\infty$

$$\mu = \prod_{k=1}^{\infty} \mu_k, \ \nu = \prod_{k=1}^{\infty} \nu_k,$$

and the corresponding Hellinger integral $H(\mu, \nu)$. As is easily checked we have

$$H(\mu, \nu) = \prod_{k=1}^{\infty} H(\mu_k, \nu_k).$$

**Proposition 1.3.5 (Kakutani)** If $H(\mu, \nu) > 0$ then $\mu$ and $\nu$ are equivalent. Moreover

$$f(x) := \frac{d\nu}{d\mu}(x) = \prod_{k=1}^{\infty} \frac{d\nu_k}{d\mu_k}(x_k), \ x \in \mathbb{R}^\infty, \ \mu\text{-a.e.} \quad (1.3.5)$$

**Proof.** We set

$$f_n(x) = \prod_{k=1}^{n} \frac{d\nu_k}{d\mu_k}(x_k), \ x \in \mathbb{R}^\infty, \ n \in \mathbb{N}.$$
Gaussian measures

We are going to prove that the sequence \((f_n)\) is convergent on \(L^1(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)\). Let \(m, n \in \mathbb{N}\), then we have

\[
\int_{\mathbb{R}^\infty} \left| f_n^{1/2}(x) - f_n^{1/2}(x) \right|^2 \mu(dx)
\]

\[
= \int_{\mathbb{R}^\infty} \prod_{k=1}^n \frac{d\nu_k}{d\mu_k}(x_k) \left| \prod_{k=n+1}^{n+m} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx)
\]

\[
= \prod_{k=1}^n \int_{\mathbb{R}^\infty} \frac{d\nu_k}{d\mu_k}(x_k) \mu(dx) \int_{\mathbb{R}^\infty} \left| \prod_{k=n+1}^{n+m} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} - 1 \right|^2 \mu(dx).
\]

Consequently

\[
\int_{\mathbb{R}^\infty} \left| f_n^{1/2+p}(x) - f_n^{1/2+p}(x) \right|^2 \mu(dx)
\]

\[
= \int_{\mathbb{R}^\infty} \left[ \prod_{k=n+1}^{n+p} \frac{d\nu_k}{d\mu_k}(x_k) - 2 \prod_{k=n+1}^{n+p} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} + 1 \right] \mu(dx)
\]

\[
= 2 \left( 1 - \prod_{k=n+1}^{n+p} \int_{\mathbb{R}} \left( \frac{d\nu_k}{d\mu_k}(x_k) \right)^{1/2} \mu_k(dx_k) \right)
\]

\[
= 2 \left( 1 - \prod_{k=n+1}^{n+p} H(\mu_k, \nu_k) \right).
\]

(1.3.6)

On the other hand we know by assumption that

\[ H(\mu, \nu) = \prod_{k=1}^\infty H(\mu_k, \nu_k) > 0, \]

or, equivalently, that

\[ - \log H(\mu, \nu) = - \sum_{k=1}^\infty \log[H(\mu_k, \nu_k)] < +\infty. \]
Consequently, for any $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that if $n > n_\varepsilon$ and $p \in \mathbb{N}$, we have

$$- \sum_{k=n_\varepsilon+1}^{n+p} \log[H(\mu_k, \nu_k)] < \varepsilon.$$ 

By (1.3.6) if $n > n_\varepsilon$ we have

$$\int_{\mathbb{R}} |\sqrt{f_{n+p}} - \sqrt{f_n}|^2 d\mu \leq 2(1 - e^{-\varepsilon}).$$

Thus the sequence $(f_n^{1/2})$ is convergent on $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ to some function $f^{1/2}$. Therefore $f_n \to f$ in $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$.

Finally, we prove that $\nu << \mu$ and $f = \frac{d\nu}{d\mu}$. Let $\varphi$ be a continuous bounded Borel function on $\mathbb{R}^\infty$, and set $\varphi_n(x) = \varphi(P_n(x))$, $x \in \mathbb{R}^\infty$, where $P_n x = \{x_1, \ldots, x_n, 0, 0, \ldots\}$. Then we have

$$\int_{\mathbb{R}^\infty} \varphi(P_n x) \nu(dx) = \int_{\mathbb{R}^n} \varphi(P_n x) \nu_1(dx_1) \ldots \nu_n(dx_n)$$

$$= \int_{\mathbb{R}^n} \varphi(P_n x) \frac{d\nu_1}{d\mu_1}(x_1) \ldots \frac{d\nu_n}{d\mu_n}(x_n) \mu_1(dx_1) \ldots \mu_n(dx_n)$$

$$= \int_{\mathbb{R}^\infty} \varphi(P_n x) f_n(x) \mu(dx).$$

Letting $n$ tend to infinity, we find

$$\int_{\mathbb{R}^\infty} \varphi(x) \nu(dx) = \int_{\mathbb{R}^\infty} \varphi(x) f(x) \mu(dx),$$

so that $\nu << \mu$. Finally, by exchanging the rôles of $\mu$ and $\nu$, we find $\mu << \nu$. 

1.3.2 The Cameron-Martin formula

We consider here the measures $\mu = N_{a,Q}$ and $\nu = N_Q$, and for any $a \in Q^{1/2}(H)$ we set

$$\rho_a(x) = \exp \left\{ - \frac{1}{2} |Q^{-1/2}a|^2 + \langle Q^{-1/2}a, Q^{-1/2}x \rangle \right\}, \ x \in H. \tag{1.3.7}$$

Let us recall, see §1.2.4, that $W_f(x) = \langle f, Q^{-1/2}x \rangle$ was defined for all $f \in Q^{1/2}(H)$. Since $Q^{-1/2}a \in Q^{1/2}(H)$ the definition (1.3.7) is meaningful.