Gauge Field Theories

Second Edition

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1

Classical fields, symmetries and their breaking

Classically, we distinguish particles and forces which are responsible for interaction between particles. The forces are described by classical fields. The motion of particles in force fields is subject to the laws of classical mechanics. Quantization converts classical mechanics into quantum mechanics which describes the behaviour of particles at the quantum level. A state of a particle is described by a vector $|\Phi(t)\rangle$ in the Hilbert space or by its concrete representation, e.g. the wave-function $\Phi(x) = \langle \mathbf{x}|\Phi(t)\rangle$ $(x = (t, \mathbf{x}))$ whose modulus squared is interpreted as the density of probability of finding the particle in point $\mathbf{x}$ at the time t. As a next step, the wave-function is interpreted as a physical field and quantized. We then arrive at quantum field theory as the universal physical scheme for fundamental interactions. It is also reached by quantization of fields describing classical forces, such as electromagnetic forces. The basic physical concept which underlies quantum field theory is the equivalence of particles and forces. This logical structure of theories for fundamental interactions is illustrated by the diagram shown below:

PARTICLES	FORCES
Classical mechanics (quantization)	Classical field theory (quantization)
Quantum mechanics (wave-function $\Phi(x)$ is interpreted as a physical field and quantized)	

QUANTUM FIELD THEORY

This chapter is devoted to a brief summary of classical field theory. The reader should not be surprised by such formulations as, for example, the 'classical' Dirac field which describes the spin $\frac{1}{2}$ particle. It is interpreted as a wave function for the Dirac particle. Our ultimate goal is quantum field theory and our classical fields in this chapter are not necessarily only the fields which describe the classical forces observed in Nature.

1.1 The action, equations of motion, symmetries and conservation laws

Equations of motion

All fundamental laws of physics can be understood in terms of a mathematical construct: *the action*. An ansatz for the action $S = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}$ can be regarded as a formulation of a theory. In classical field theory the lagrangian density $\mathcal{L}$ is a function of fields Φ and their derivatives. In general, the fields Φ are multiplets under Lorentz transformations and in a space of internal degrees of freedom. It is our experience so far that in physically relevant theories the action satisfies several general principles such as: (i) Poincaré invariance (or general covariance for theories which take gravity into consideration), (ii) locality of $\mathcal{L}$ and its, at most, bilinearity in the derivatives $\partial_\nu\Phi(x)$ (to get at most second order differential equations of motion),† (iii) invariance under all symmetry transformations which characterize the considered physical system, (iv) S has to be real to account for the absence of absorption in classical physics and for conservation of probability in quantum physics. The action is then the most general functional which satisfies the above constraints.

From the action we get:

equations of motion (invoking Hamilton's principle);
conservations laws (from Noether's theorem);
transition from classical to quantum physics (by using path integrals or canonical quantization).

In this section we briefly recall the first two results. Path integrals are discussed in detail in Chapter 2.

The equations of motion for a system described by the action

$$S = \int_{t_0}^{t_1} \mathrm{d}t \int_V \mathrm{d}^3x\, \mathcal{L}\left(\Phi(x), \partial_\mu\Phi(x)\right) \tag{1.1}$$

(the volume V may be finite or infinite; in the latter case we assume the system to be localized in space‡) are obtained from Hamilton's principle. This ansatz (whose physical sense becomes clearer when classical theory is considered as a limit of quantum theory formulated in terms of path integrals) says: the dynamics of the system evolving from the initial state $\Phi(t_0, \mathbf{x})$ to the final state $\Phi(t_1, \mathbf{x})$ is

† Later we shall encounter effective lagrangians that may contain terms with more derivatives. Such terms can usually be interpreted as a perturbation.

‡ Although we shall often use plane wave solutions to the equations of motion, we always assume implicitly that real physical systems are described by wave packets localized in space.

such that the action (taken as a functional of the fields and their first derivatives) remains stationary during the evolution, i.e.

$$\delta S = 0 \tag{1.2}$$

for arbitrary variations

$$\Phi(x) \to \Phi(x) + \delta\Phi(x) \tag{1.3}$$

which vanish on the boundary of $\Omega \equiv (\Delta t, V)$.

Explicit calculation of the variation δS gives:

$$\delta S = \int_\Omega \mathrm{d}^4 x \left[\frac{\partial \mathcal{L}}{\partial \Phi} \delta\Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \delta(\partial_\nu \Phi) \right] \tag{1.4}$$

(summation over all fields Φ and for each field Φ over its Lorentz and 'internal' indices is always understood). Since in the variation the coordinates x do not change we have:

$$\delta(\partial_\nu \Phi) = \partial_\nu \delta\Phi \tag{1.5}$$

and (1.4) can be rewritten as follows:

$$\delta S = \int_\Omega \mathrm{d}^4 x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \right] \delta\Phi + \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} \delta\Phi \right] \right\} \tag{1.6}$$

The second term in (1.6) is a surface term which vanishes and, therefore, the condition $\delta S = 0$ gives us the Euler–Lagrange equations of motion for the classical fields:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i^\mu} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi_i^\mu)} = 0 \qquad \begin{array}{l} \nu,\ \mu = 0, 1, 2, 3 \\ i = 1, \ldots, n \end{array} \tag{1.7}$$

(here we keep the indices explicitly, with Φ taken as a Lorentz vector; is are internal quantum number indices). It is important to notice that lagrangian densities which differ from each other by a total derivative of an arbitrary function of fields

$$\mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu(\Phi) \tag{1.8}$$

give the same classical equations of motion (due to the vanishing of variations of fields on the boundary of Ω).

Global symmetries

Eq. (1.6) can also be used to derive conservation laws which follow from a certain class of symmetries of the physical system. Let us consider a Lie group of continuous global (x-independent) infinitesimal transformations (here we restrict ourselves to unitary transformations; conformal transformations are discussed in

Chapter 7) acting in the space of internal degrees of freedom of the fields Φ_i. Under infinitesimal rotation of the reference frame (passive view) or of the physical system (active view) in that space:

$$\Phi_i(x) \to \Phi'_i(x) = \Phi_i(x) + \delta_0\Phi_i(x) \tag{1.9}$$

where

$$\delta_0\Phi_i(x) = -\mathrm{i}\Theta^a T^a_{ij}\Phi_j(x) \tag{1.10}$$

and $\Phi'_i(x)$ is understood as the ith component of the field $\Phi(x)$ in the new reference frame or of the rotated field in the original frame (active). Note that a rotation of the reference frame by angle Θ corresponds to active rotation by $(-\Theta)$. The T^as are a set of hermitean matrices T^a_{ij} satisfying the Lie algebra of the group G

$$\left[T^a, T^b\right] = \mathrm{i}c^{abc}T^c, \qquad \mathrm{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab} \tag{1.11}$$

and the Θ^as are x-independent.

Under the change of variables $\Phi \to \Phi'(x)$ given by (1.9) the lagrangian density is transformed into

$$\mathcal{L}'\big(\Phi'(x), \partial_\mu\Phi'(x)\big) \equiv \mathcal{L}\big(\Phi(x), \partial_\mu\Phi(x)\big) \tag{1.12}$$

and, of course, the action remains numerically unchanged:

$$\delta_\Theta S = S' - S = \int_\Omega \mathrm{d}^4x \left[\mathcal{L}'(\Phi', \partial_\mu\Phi') - \mathcal{L}(\Phi, \partial_\mu\Phi)\right] = 0 \tag{1.13}$$

(we use the symbol $\delta_\Theta S$ for the change of the action under transformation (1.9) on the fields to distinguish it from the variations $\delta S'$ and δS). For the variations we have $\delta S' = \delta S$. Thus, if $\Phi(x)$ describes a motion of a system, $\Phi'(x)$ is also a solution of the equations of motion for the transformed fields which, however, in general are different in form from the original ones.

Transformations (1.9) are symmetry transformations for a physical system if its equations of motion remain form-invariant in the transformed fields. In other words, a solution to the equations of motion after being transformed according to (1.9) remains a solution of the same equations. This is ensured if the density $\mathcal{L}$ is invariant under transformations (1.9) (for scale transformations see Chapter 7):

$$\mathcal{L}'(\Phi'(x), \partial_\mu\Phi'(x)) = \mathcal{L}(\Phi'(x), \partial_\mu\Phi'(x)) \tag{1.14}$$

or, equivalently,

$$\mathcal{L}(\Phi'(x), \partial_\mu\Phi'(x)) = \mathcal{L}(\Phi(x), \partial_\mu\Phi(x)).$$

Indeed, variation of the action generated by arbitrary variations of the fields $\Phi'(x)$ with boundary conditions (1.3) is again given by (1.6) (with $\Phi \to \Phi'$ etc.)

and we derive the same equations for $\Phi'(x)$ as for $\Phi(x)$. Moreover, the change (1.14) of the lagrangian density under the symmetry transformations:

$$\mathcal{L}'(\Phi', \partial_\mu \Phi') - \mathcal{L}(\Phi, \partial_\mu \Phi) = \mathcal{L}(\Phi', \partial_\mu \Phi') - \mathcal{L}(\Phi, \partial_\mu \Phi) = 0 \tag{1.15}$$

is formally given by the integrand in (1.6) with $\delta\Phi$s given by (1.9). It follows from (1.15) and the equations of motion that the currents

$$j^a_\mu(x) = -\mathrm{i}\frac{\partial \mathcal{L}}{\partial(\partial^\mu \Phi_i)} T^a_{ij} \Phi_j \tag{1.16}$$

are conserved and the charges

$$Q^a(t) = \int \mathrm{d}^3 x \; j^a_0(t, \mathbf{x}) \tag{1.17}$$

are constants of motion, provided the currents fall off sufficiently rapidly at the space boundary of Ω.

It is very important to notice that the charges defined by (1.17) (even if they are not conserved) are the generators of the transformation (1.10). Firstly, they satisfy (in an obvious way) the same commutation relation as the matrices T^a. Secondly, they indeed generate transformations (1.10) through Poisson brackets. Let us introduce conjugate momenta for the fields $\Phi(x)$:

$$\Pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Phi(t, \mathbf{x}))} \tag{1.18}$$

(we assume here that $\Pi \neq 0$; there are interesting exceptions, with $\Pi = 0$, such as, for example, classical electrodynamics; these have to be discussed separately). Again, the Lorentz and 'internal space' indices of the Πs and Φs are hidden. The hamiltonian of the system reads

$$H = \int \mathrm{d}^3 x \, \mathcal{H}(\Pi, \Phi), \qquad \mathcal{H} = \Pi \partial_0 \Phi - \mathcal{L} \tag{1.19}$$

and the Poisson bracket of two functionals F_1 and F_2 of the fields Π and Φ is defined as

$$\{F_1(t, \mathbf{x}), F_2(t, \mathbf{y})\} = \int \mathrm{d}^3 z \left[\frac{\partial F_1(t, \mathbf{x})}{\partial \Phi(t, \mathbf{z})} \frac{\partial F_2(t, \mathbf{y})}{\partial \Pi(t, \mathbf{z})} - \frac{\partial F_1(t, \mathbf{x})}{\partial \Pi(t, \mathbf{z})} \frac{\partial F_2(t, \mathbf{y})}{\partial \Phi(t, \mathbf{z})} \right] \tag{1.20}$$

We get, in particular,

$$\left. \begin{array}{l} \{\Pi(t, \mathbf{x}), \Phi(t, \mathbf{y})\} = -\delta(\mathbf{x} - \mathbf{y}) \\ \{\Pi(t, \mathbf{x}), \Pi(t, \mathbf{y})\} = \{\Phi(t, \mathbf{x}), \Phi(t, \mathbf{y})\} = 0 \end{array} \right\} \tag{1.21}$$

The charge (1.17) can be written as

$$Q^a(t) = \int \mathrm{d}^3 x \, \Pi(t, \mathbf{x})(-\mathrm{i}T^a)\Phi(t, \mathbf{x}) \tag{1.22}$$

and for its Poisson bracket with the field Φ we get:

$$\left\{Q^a(t), \Phi(t, \mathbf{x})\right\} = (\mathrm{i}T^a)\Phi(t, \mathbf{x}) \tag{1.23}$$

i.e. indeed the transformation (1.10). Thus, generators of symmetry transformations are conserved charges and vice versa.

Space-time transformations

Finally, we consider transformations (changes of reference frame) which act simultaneously on the coordinates and the fields. The well-known examples are, for instance, translations, spatial rotations and Lorentz boosts. In this case our formalism has to be slightly generalized. We consider an infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \delta x^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \tag{1.24}$$

and suppose that the fields $\Phi(x)$ seen in the first frame transform under some representation $T(\varepsilon)$ of the transformation (1.24) (which acts on their Lorentz structure) into $\Phi'(x')$ in the transformed frame:

$$\Phi'(x') = \exp[-\mathrm{i}T(\varepsilon)]\Phi(x) \tag{1.25}$$

(the Lorentz indices are implicit and $\Phi'_{\alpha}(x')$ is understood as the αth component of the field $\Phi(x')$ in the new reference frame). For infinitesimal transformations we have (∂' denotes differentiation with respect to x'):

$$\left.\begin{aligned} \Phi'(x') &= \Phi'(x) + \delta x^{\mu}\partial_{\mu}\Phi(x) + \mathcal{O}(\varepsilon^2) \\ \partial'_{\mu}\Phi'(x') &= \partial_{\mu}\Phi'(x) + \delta x^{\nu}\partial_{\mu}\partial_{\nu}\Phi(x) + \mathcal{O}(\varepsilon^2) \end{aligned}\right\} \tag{1.26}$$

The lagrangian density $\mathcal{L}'$ after the transformation is defined by the equation (see, for example, Trautman (1962, 1996))

$$\begin{aligned} S' &= \int_{\Omega'} \mathrm{d}^4x'\, \mathcal{L}'\left(\Phi'(x'), \partial'_{\mu}\Phi'(x')\right) \\ &= \int_{\Omega} \mathrm{d}^4x \left[\mathcal{L}\left(\Phi(x), \partial_{\mu}\Phi(x)\right) - \partial_{\mu}\delta\Lambda^{\mu}\left(\Phi(x)\right)\right] \end{aligned} \tag{1.27}$$

(Ω' denotes the image of Ω under the transformation (1.24) and $\delta\Lambda^{\mu}$ is an arbitrary function of the fields) which is a sufficient condition for the new equations of motion to be equivalent to the old ones. By this we mean that a change of the reference frame has no implications for the motion of the system, i.e. a transformed solution to the original equations remains a solution to the new equations (since (1.27) implies $\delta S' = \delta S$). In general $\det(\partial x'/\partial x) \neq 1$ and also $\mathcal{L}'\big(\Phi'(x), \partial'_{\mu}\Phi'(x')\big) \neq \mathcal{L}\big(\Phi(x), \partial_{\mu}\Phi(x)\big)$. Moreover, there is an arbitrariness in the choice of $\mathcal{L}'$ due to the presence in (1.27) of the total derivative of an arbitrary function $\delta\Lambda^\mu(\Phi(x))$.

Symmetry transformations are again defined as transformations which leave equations of motion form-invariant. A sufficient condition is that, for a certain choice of $\delta\Lambda$, the $\mathcal{L}'$ defined by (1.27) satisfies the equation:

$$\mathcal{L}'(\Phi'(x'), \partial'_\mu \Phi'(x')) = \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x')) \tag{1.28}$$

or, equivalently,

$$\mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x'))\, \mathrm{d}^4x' = \left[\mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) - \partial_\mu \delta\Lambda^\mu(\Phi(x))\right] \mathrm{d}^4x \tag{1.28a}$$

Note that, if $\det(\partial x'/\partial x) = 1$, we recover condition (1.14) up to the total derivative. The most famous example of symmetry transformations up to a non-vanishing total derivative is supersymmetry (see Chapter 15).

The change of the action under symmetry transformations can be calculated in terms of

$$\delta\mathcal{L} = \mathcal{L}(\Phi'(x'), \partial'_\mu \Phi'(x')) - \mathcal{L}(\Phi(x), \partial_\mu \Phi(x)) \tag{1.29}$$

where x and x' are connected by the transformation (1.24). Using (1.26) we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi}\delta_0\Phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0(\partial_\mu\Phi) + \delta x^\mu \partial_\mu\mathcal{L} + \mathcal{O}(\varepsilon^2) \tag{1.30}$$

where

$$\delta_0\Phi(x) = \Phi'(x) - \Phi(x) \tag{1.31}$$

and, therefore, since δ_0 is a functional change,

$$\delta_0\partial_\mu\Phi = \partial_\mu\delta_0\Phi \tag{1.32}$$

Eq. (1.30) can be rewritten in the following form:

$$\delta\mathcal{L} = \delta x^\mu \partial_\mu\mathcal{L} + \left[\frac{\partial\mathcal{L}}{\partial\Phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\right]\delta_0\Phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi\right) \tag{1.33}$$

Since

$$\det\left(\frac{\partial x'_\mu}{\partial x_\nu}\right) = 1 + \partial_\mu\delta x^\mu \tag{1.34}$$

we finally get (using (1.28a), (1.33) and the equations of motion):

$$\begin{aligned} 0 &= \mathcal{L}\partial_\mu\delta x^\mu + \delta x^\mu\partial_\mu\mathcal{L} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi\right) + \partial_\mu\delta\Lambda^\mu + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{L}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\delta_0\Phi + \delta\Lambda^\mu + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{1.35}$$

Re-expressing $\delta_0\Phi$ in terms of $\delta\Phi = \Phi'(x') - \Phi(x) = \delta_0\Phi + \delta x^\mu \partial_\mu \Phi$ we conclude that

$$\partial_\mu j^\mu \equiv \partial_\mu \left\{ \left[\mathcal{L} g^\mu{}_\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\rho \Phi \right] \delta x^\rho + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \delta\Phi + \delta\Lambda^\mu \right\} = 0 \quad (1.36)$$

This is a generalization of the result (1.16) to space-time symmetries and both results are the content of the Noether's theorem which states that symmetries of a physical system imply conservation laws.

Examples

We now consider several examples. Invariance of a physical system under translations

$$x'^\mu = x^\mu + \varepsilon^\mu \quad (1.37)$$

implies the conservation law

$$\partial_\mu \Theta^{\mu\nu}(x) = 0 \quad (1.38)$$

where $\Theta^{\mu\nu}$ is the energy–momentum tensor

$$\Theta^{\mu\nu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial^\nu \Phi - g^{\mu\nu} \mathcal{L} \quad (1.39)$$

(we have assumed that the considered system is such that $\mathcal{L}'(x')$ defined by (1.27) with $\Lambda \equiv 0$ satisfies $\mathcal{L}'(x') = \mathcal{L}(x')$; note also that $\det(\partial x'/\partial x) = 1$). The four constants of motion

$$P^\nu = \int \mathrm{d}^3 x \, \Theta^{0\nu}(x) \quad (1.40)$$

are the total energy of the system ($\nu = 0$) and its momentum vector ($\nu = 1, 2, 3$). It is easy to check that transformations of the field $\Phi(x)$ under translations in time and space are given by

$$\{P^\mu, \Phi(x)\} = -\frac{\partial}{\partial x_\mu} \Phi(x) \quad (1.41)$$

so that $\Phi'(x) = \Phi(x) + \{P^\mu, \Phi(x)\} \, \varepsilon_\mu$.

The energy–momentum tensor defined by (1.39) is the so-called canonical energy–momentum tensor. It is important to notice that the physical interpretation of the energy–momentum tensor remains unchanged under the redefinition

$$\Theta^{\mu\nu}(x) \to \Theta^{\mu\nu}(x) + f^{\mu\nu}(x) \quad (1.42)$$

where $f^{\mu\nu}(x)$ are arbitrary functions satisfying the conservation law

$$\partial_\mu f^{\mu\nu}(x) = 0 \quad (1.43)$$

with vanishing charges

$$\int \mathrm{d}^3x \; f^{0\nu}(x) = 0 \tag{1.44}$$

Then (1.38) and (1.40) remain unchanged. A solution to these constraints is

$$f^{\mu\nu}(x) = \partial_\rho f^{\mu\nu\rho} \tag{1.45}$$

where $f^{\mu\nu\rho}(x)$ is antisymmetric in indices $(\mu\rho)$. This freedom can be used to make the energy–momentum tensor symmetric and gauge-invariant (see Problem 1.3).

An infinitesimal Lorentz transformation reads

$$x'^\mu \equiv \Lambda^\mu{}_\nu(\omega)x^\nu \approx x^\mu + \omega^\mu{}_\nu x^\nu \tag{1.46}$$

(with $\omega_{\mu\nu}$ antisymmetric) and it is accompanied by the transformation on the fields

$$\Phi'(x') = \exp\left(\tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Phi(x) \approx \left(1 + \tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right)\Phi(x) \tag{1.47}$$

where $\Sigma^{\mu\nu}$ is a spin matrix. For Lorentz scalars $\Sigma = 0$, for Dirac spinors $\Sigma^{\mu\nu} = (-\mathrm{i}/2)\sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ (for the notation see Appendix A). Inserting (1.46) and (1.47) into (1.36) (using (1.38) and replacing the canonical energy–momentum tensor by a symmetric one and finally using the antisymmetry of $\omega^{\mu\nu}$), we get the following conserved currents:

for a scalar field:

$$M^{\mu,\nu\rho}(x) = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu} \tag{1.48}$$

for a field with a non-zero spin:

$$M^{\mu,\nu\rho}(x) = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)}\Sigma^{\nu\rho}\Phi \tag{1.49}$$

The conserved charges

$$M^{\nu\rho}(t) = \int \mathrm{d}^3x \, M^{0,\nu\rho}(t, \mathbf{x}) \tag{1.50}$$

are identified with the total angular momentum tensor of the considered physical system. The charges $M^{\mu\nu}$ are the generators of Lorentz transformations on the fields $\Phi(x)$ with

$$\{M^{\mu\nu}, \Phi(x)\} = -(x^\mu\partial^\nu - x^\nu\partial^\mu + \Sigma^{\mu\nu})\Phi(x) \tag{1.51}$$

so that $\Phi'(x) = \Phi(x) - \frac{1}{2}\omega_{\mu\nu}\{M^{\mu\nu}, \Phi(x)\}$.

The tensor $M^{\mu\nu}$ is not translationally invariant. The total angular momentum three-vector is obtained by constructing the Pauli–Lubański vector

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\kappa}M^{\nu\rho}P^\kappa \tag{1.52}$$

(where $\varepsilon_{\mu\nu\rho\kappa}$ is the totally antisymmetric tensor defined by $\varepsilon_{0123} = -\varepsilon^{0123} = 1$) which reduces in the rest frame $P^\mu = (m, \mathbf{0})$ to the three-dimensional total angular momentum $M^k \equiv \frac{1}{2}\varepsilon^{ijk}M^{ij}$: $M^k = W^k/m$.

1.2 Classical field equations

Scalar field theory and spontaneous breaking of global symmetries

In this section we illustrate our general considerations with the simplest example: the classical theory of scalar fields $\Phi(x)$. We take these to be complex, in order to be able to discuss a theory which is invariant under continuous global symmetry of phase transformations ($U(1)$ symmetry group of one-parameter, unitary, unimodular transformations):

$$\Phi'(x) = \exp(-\mathrm{i}q\Theta)\Phi(x) \approx (1 - \mathrm{i}q\Theta)\Phi(x) \tag{1.53}$$

Thus, the field $\Phi(x)$ (its complex conjugate $\Phi^\star(x)$) carries the internal quantum number q $(-q)$. It is sometimes also useful to introduce two real fields

$$\phi = \frac{1}{\sqrt{2}}(\Phi + \Phi^\star), \qquad \chi = \frac{-\mathrm{i}}{\sqrt{2}}(\Phi - \Phi^\star) \tag{1.54}$$

The lagrangian density which satisfies all the constraints of the previous section reads:

$$\mathcal{L} = \partial_\mu\Phi\partial^\mu\Phi^\star - m^2\Phi\Phi^\star - \frac{1}{2}\lambda(\Phi\Phi^\star)^2 - \frac{1}{3m_1^2}(\Phi\Phi^\star)^3 + \cdots = T - V \tag{1.55}$$

where T is the kinetic energy density and V is the potential energy density. The terms which are higher than second powers of the bilinear combination $\Phi\Phi^\star$ require on dimensional grounds couplings with dimension of negative powers of mass.†

Using the formalism of the previous section we derive the equation of motion (since we have two physical degrees of freedom, the variations in Φ and $\Phi^\star$ are independent):

$$(\Box + m^2)\Phi + \lambda\Phi\Phi^\star\Phi + \frac{1}{m_1^2}(\Phi\Phi^\star)^2\Phi + \cdots = 0 \tag{1.56}$$

In the limit of vanishing couplings (free fields) we get the Klein–Gordon equation. The conserved $U(1)$ current

$$j_\mu(x) = \mathrm{i}q(\Phi^\star\partial_\mu\Phi - \Phi\partial_\mu\Phi^\star) \tag{1.57}$$

† In units $c = 1$, the action has the dimension $[S] = \mathrm{g}\cdot\mathrm{cm} = [\hbar]$. Thus $[\mathcal{L}] = \mathrm{g}\cdot\mathrm{cm}^{-3}$, $[\Phi] = (\mathrm{g/cm})^{1/2}$ and the coefficients $[m^2] = \mathrm{cm}^{-2}$, $[\lambda] = (\mathrm{g}\cdot\mathrm{cm})^{-1}$, $[m_1^2] = \mathrm{g}^2$. In units $c = \hbar = 1$ we have $[\mathcal{L}] = \mathrm{g}^4$, $[\Phi] = \mathrm{g}$, $[m^2] = [m_1^2] = \mathrm{g}^2$ and S and λ are dimensionless.

and the canonical energy–momentum tensor

$$\Theta_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi^\star + \partial_\nu \Phi \partial_\mu \Phi^\star - g_{\mu\nu}(\partial_\rho \Phi \partial^\rho \Phi^\star) \\ + g_{\mu\nu} m^2 \Phi\Phi^\star + \tfrac{1}{2}\lambda g_{\mu\nu}(\Phi\Phi^\star)^2 + \cdots \tag{1.58}$$

are obtained from (1.16) and (1.33), respectively. This tensor can be written in other forms, for example, by adding the term $\partial_\rho(g_{\mu\nu}\Phi\partial^\rho\Phi^\star - g_{\rho\nu}\Phi\partial^\mu\Phi^\star)$ and using the equation of motion. Invariance under Lorentz transformation gives the conserved current (1.48).

The $U(1)$ symmetry of our scalar field theory can be realized in the so-called Wigner mode or in the Goldstone–Nambu mode, i.e. it can be spontaneously broken. The phenomenon of spontaneous breaking of global symmetries is well known in condensed matter physics. The standard example of a physical theory with spontaneous symmetry breakdown is the Heisenberg ferromagnet, an infinite array of spin $\frac{1}{2}$ magnetic dipoles. The spin–spin interactions between neighbouring dipoles cause them to align. The hamiltonian is rotationally invariant but the ground state is not: it is a state in which all the dipoles are aligned in some arbitrary direction. So for an infinite ferromagnet there is an infinite degeneracy of the vacuum.

We say that the symmetry G is spontaneously broken if the ground state (usually called the vacuum) of a physical system with symmetry G (i.e. described by a lagrangian which is symmetric under G) is not invariant under the transformations G. Intuitively speaking, the vacuum is filled with scalars (with zero four-momenta) carrying the quantum number (s) of the broken symmetry. Since we do not want spontaneously to break the Lorentz invariance, spontaneous symmetry breaking can be realized only in the presence of scalar fields (fundamental or composite).

Let us return to the question of spontaneous breaking of the $U(1)$ symmetry in the theory defined by the lagrangian (1.55). The minimum of the potential occurs for the classical field configurations such that

$$\frac{\partial V}{\partial \Phi} = \frac{\partial V}{\partial \Phi^\star} = 0 \tag{1.59}$$

For $m^2 > 0$ the minimum of the potential exists for $\Phi = \Phi_0 = 0$ and $V(\Phi_0) = 0$. However, for $m^2 < 0$ (and $m_1^2 > 0$; study the potential for $m_1^2 < 0$!) the solutions to these equations are (note that λ must be positive for the potential to be bounded from below)

$$\Phi_0(\Theta) = \exp(\mathrm{i}\Theta)\left(\frac{-m^2}{\lambda}\right)^{1/2}\left[1 + \mathcal{O}\left(\frac{m^2}{\lambda^2 m_1^2}\right)\right] \tag{1.60}$$

with $V(\Phi_0) = -(m^4/2\lambda)$. The dependence of the potential on the field Φ is shown in Fig. 1.1. Thus, the $U(1)$ symmetric state is no longer the ground state of this

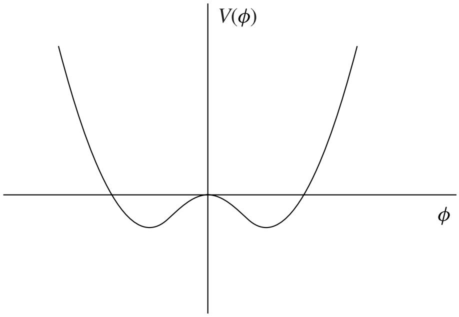

Fig. 1.1.

theory. Instead, the potential has its minimum for an infinite, degenerate set of states (one often calls it a 'flat direction') connected to each other by the $U(1)$ rotations. In this example we can choose any one of these vacua as the ground state of our theory with spontaneously broken $U(1)$ symmetry.

Physical interpretation of this model (and of the negative mass parameter m^2) is obtained if we expand the lagrangian (1.55) around the true ground state, i.e. rewrite it in terms of the fields (we choose $\Theta = 0$ and put $m_1^2 = \infty$): $\tilde{\Phi}(x) = \Phi(x) - \Phi_0$. After a short calculation we see that the two dynamical degrees of freedom, $\tilde{\phi}(x)$ and $\tilde{\chi}(x)$ (see (1.54)), have the mass parameters $(-m^2)$ (i.e. positive) and zero, respectively (see Chapter 11 for more details). The $\tilde{\chi}(x)$ describes the so-called Goldstone boson – a massless mode whose presence is related to the spontaneous breaking of a continuous global symmetry. The degenerate set of vacua (1.60) requires massless excitations to transform one vacuum into another one. We shall return to this subject later (Chapter 9).

In theories with spontaneously broken symmetries the currents remain conserved but the charges corresponding to the broken generators of the symmetry group are only formal in the sense that, generically, the integral (1.17) taken over all space does not exist. Still, the Poisson brackets (or the commutators) of a charge with a field can be defined provided we integrate over the space after performing the operation at the level of the current (see, for example, Bernstein (1974)).

Spinor fields

The basic objects which describe spin $\frac{1}{2}$ particles are two-component anticommuting Weyl spinors λ_α and $\bar{\chi}^{\dot{\alpha}}$, transforming, respectively, as representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group or, in other words, as two inequivalent representations of the $SL(2, C)$ group of complex two-dimensional matrices M

of determinant 1:

$$\lambda'_\alpha(x') = M_\alpha{}^\beta \lambda_\beta(x) \tag{1.61}$$

$$\bar{\chi}'^{\dot{\alpha}}(x') = \left(M^{\dagger -1}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{\chi}^{\dot{\beta}}(x) \tag{1.62}$$

where x' and x are connected by the transformation (1.46),

$$M = \exp\left[-(\mathrm{i}/4)\omega_{\mu\nu}\sigma^{\mu\nu}\right] \tag{1.63}$$

and the two-dimensional matrices $\sigma^{\mu\nu}$ are defined in Appendix A. We use the same symbol for two- and four-dimensional matrices $\sigma^{\mu\nu}$ since it should not lead to any confusion in every concrete context. Two other possible representations of the $SL(2, C)$ group denoted as λ^α and $\bar{\chi}_{\dot{\alpha}}$ and transforming through matrices $(M^{-1})^T$ and M^* are unitarily equivalent to the representations (1.61) and (1.62), respectively. The corresponding unitary transformations are given by:

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \qquad \bar{\chi}^{\dot{\alpha}} = \bar{\chi}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}} \tag{1.64}$$

where the antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ are defined in Appendix A. Since transition from the representation (1.61) to (1.62) involves complex conjugation it is natural to denote one of them with a bar. The distinction between the undotted and dotted indices is made to stress that the two representations are inequivalent. The pairwise equivalent representations correspond to lower and upper indices. The choice of (1.61) and (1.62) as fundamental representations is dictated by our convention which identifies (A.32) with the matrix M in (A.16). Those unfamiliar with two-component spinors and their properties should read Appendix A before starting this subsection.

Scalars of $SL(2, C)$ can now be constructed as antisymmetric products $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$ and $(0, \frac{1}{2}) \otimes (0, \frac{1}{2})$. They have, for example, the structure $\varepsilon^{\alpha\beta}\lambda_\beta\phi_\alpha = \lambda^\alpha\phi_\alpha = -\lambda_\alpha\phi^\alpha \equiv \lambda \cdot \phi$ or $\bar{\chi}^{\dot{\beta}}\bar{\eta}^{\dot{\alpha}}\varepsilon_{\dot{\beta}\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}}\eta^{\dot{\alpha}} = -\bar{\chi}^{\dot{\alpha}}\bar{\eta}_{\dot{\alpha}} \equiv \bar{\chi} \cdot \bar{\eta}$. Thus, the tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$ are metric tensors for the spinor representations. A four-vector is a product of $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations, for example, $\varepsilon^{\alpha\beta}\lambda_\beta(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\chi}^{\dot{\beta}} = \lambda^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\chi}^{\dot{\beta}}$ or $\bar{\chi}_{\dot{\alpha}}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,\lambda_\beta$ (see Appendix A for the definitions of σ^μ and $\bar{\sigma}^\mu$).

We suppose now that a physical system can be described by a Weyl field (or a set of Weyl fields) λ_α which transforms as $(\frac{1}{2}, 0)$ under the Lorentz group. Its complex conjugates are $\bar{\lambda}_{\dot{\alpha}} \equiv (\lambda_\alpha)^\star$ (see the text above (1.64)) and $\bar{\lambda}^{\dot{\alpha}} = \bar{\lambda}_{\dot{\beta}}\varepsilon^{\dot{\beta}\dot{\alpha}}$ which transforms as $(0, \frac{1}{2})$ representation. (Note, that numerically $\left\{\varepsilon^{\alpha\beta}\right\} = \{\varepsilon^{\dot{\alpha}\dot{\beta}}\} = \mathrm{i}\sigma^2$.) It is clear that $SL(2, C)$-invariant kinetic terms read

$$\mathcal{L} = \mathrm{i}\lambda\sigma^\mu\partial_\mu\bar{\lambda} = \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda \tag{1.65}$$

where the two terms are equal up to a total derivative. The factor i has been introduced for hermiticity of the lagrangian since one assumes that the spinor components are anticommuting c-numbers. Indeed, this is what we have to assume for these classical fields when we follow the path integral approach to field quantization (see Chapter 2).

Furthermore, if the field λ carries no internal charges (or transforms as a real representation of an internal symmetry group), one can construct the Lorentz scalars $\lambda\lambda = \lambda^\alpha\lambda_\alpha$ and $\bar{\lambda}\bar{\lambda} = \bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$ and introduce the so-called Majorana mass term

$$\mathcal{L}_m = -\tfrac{1}{2}m\left(\lambda\lambda + \bar{\lambda}\bar{\lambda}\right) \tag{1.66}$$

The most general hermitean combination, $-\frac{1}{2}m_1(\lambda\lambda + \bar{\lambda}\bar{\lambda}) - (\mathrm{i}/2)m_2(\lambda\lambda - \bar{\lambda}\bar{\lambda})$, can be recast in the above form with $m = (m_1^2 + m_2^2)^{1/2}$ by the transformation $\lambda \to \exp(\mathrm{i}\varphi)\lambda$ with $\exp(2\mathrm{i}\varphi) = (m_1 - \mathrm{i}m_2)/(m_1^2 + m_2^2)^{1/2}$. The term $\mathcal{L}_m$ does not vanish since λ_αs are anticommuting c-numbers. Finally, we remark that a system described by a single Weyl spinor has two degrees of freedom, with the equation of motion obtained from the principle of minimal action (variations with respect to λ and $\bar{\lambda}$ are independent):

$$\left(\mathrm{i}\bar{\sigma}^\mu\partial_\mu\lambda\right)_{\dot{\alpha}} = m\bar{\lambda}_{\dot{\alpha}} \tag{1.67}$$

We now consider the case of a system described by a set of Weyl spinors λ_α^i which transform as a complex representation R (see Appendix E for the definition of complex and real representations) of a group of internal symmetry. Such theory is called chiral. Then, the kinetic lagrangian remains as in (1.65) but no invariant mass term can be constructed since Lorentz invariants $\lambda^i\lambda^j$ and $\bar{\lambda}^i\bar{\lambda}^j$ are not invariant under the symmetry group. Note that in this case the spinors $\bar{\lambda}^{\dot{\alpha}}$ (and $\bar{\lambda}_{\dot{\alpha}}$), i.e. the spinors transforming as $(0, \frac{1}{2})$ (or its unitary equivalent) under the Lorentz transformations, form the representation $R^\star$, i.e. complex conjugate to R.

There are also interesting physical systems that consist of pairs of independent sets of Weyl fields λ_α and λ_α^c (the internal symmetry indices i, for example, λ_α^i, and the summation over them are always implicit), both being $(\frac{1}{2}, 0)$ representations of the Lorentz group but with λ_α^c transforming as the representation $R^\star$, complex conjugate to R. Such theory is called vector-like. We call such spinors charge conjugate to each other. The operation of charge conjugation will be discussed in more detail in Section 1.5. The mass term can then be introduced and the full lagrangian of such a system then reads:

$$\mathcal{L} = \mathrm{i}\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda + \mathrm{i}\lambda^c\sigma^\mu\partial_\mu\bar{\lambda}^c - m\left(\lambda\lambda^c + \bar{\lambda}\bar{\lambda}^c\right) \tag{1.68}$$

Similarly to the case of Majorana mass, the mass term $-\mathrm{i}m'(\lambda\lambda^c - \bar{\lambda}\bar{\lambda}^c)$, if present, can be rotated away by the appropriate phase transformation. Remember that $\lambda^c\sigma^\mu\partial_\mu\bar{\lambda}^c = \bar{\lambda}^c\bar{\sigma}^\mu\partial_\mu\lambda^c$ (up to a total derivative), so both fields λ and λ^c enter the lagrangian in a fully symmetric way. The mass term, the so-called Dirac mass, is now invariant under the internal symmetry group. The equations of motion which follow from (1.68) are:

$$\left(\mathrm{i}\bar{\sigma}^{\mu}\partial_{\mu}\lambda\right)_{\dot{\alpha}} = m\bar{\lambda}^{c}_{\dot{\alpha}} \tag{1.69}$$

and similarly for λ^c. Thus, the mass term mixes positive energy solutions for λ with negative ones for λ^c (which are represented by positive energy solutions for $\bar{\lambda}^c_{\dot{\alpha}}$) and vice versa. In accord with the discussion of Appendix A, this can be interpreted as a mixing between different helicity states of the same massless particle.

As we shall discuss in more detail later, the electroweak theory is chiral and quantum chromodynamics is vector-like. In both cases, it is convenient to take as the fundamental fields of the theory Weyl spinors which transform as $(\frac{1}{2}, 0)$ representations of $SL(2, C)$. We recall that (as discussed in Appendix A) positive energy classical solutions for spinors λ^i_αs describe massless fermion states with helicity $-\frac{1}{2}$ and, therefore, λ^i_αs are called left-handed chiral fields. In the electroweak theory (see Chapter 12) all the left-handed physical fields are included in the set of λ^i_αs which transforms then as a reducible, complex representation R of the symmetry group $G = SU(2) \times U(1)$ ($R \neq R^\star$). To be more specific, in this case the set λ^i_α includes pairs of left-handed particles with opposite electric charges such as, for example, e^-, e^+ and quarks with electric charges $\mp\frac{1}{3}$, $\pm\frac{2}{3}$, i.e. this set forms a real representation of the electromagnetic $U(1)$ gauge group, but transforms as a complex representation R under the full $SU(2) \times U(1)$ group. Negative energy classical solutions for λ^i_αs, in the Dirac sea interpretation, describe massless helicity $+\frac{1}{2}$ states transforming as $R^\star$. Each pair of states described by λ^i_α is connected by the CP transformation (see later) and can therefore be termed particle and antiparticle. These pairs consist of the states with opposite helicities. (Whether within a given reducible representation contained in R the $-\frac{1}{2}$ or $+\frac{1}{2}$ helicity states are called particles is, of course, a matter of convention.)

An equivalent description of the same theory in terms of the fields $\bar{\lambda}^{i\dot{\alpha}}$ transforming as $(0, \frac{1}{2})$ representations of $SL(2, C)$ and as $R^\star$ under the internal symmetry group can be given. In this formulation positive (negative) energy solutions for $\bar{\lambda}^{i\dot{\alpha}}$s describe all $+\frac{1}{2}$ $(-\frac{1}{2})$ helicity massless states of the theory (formerly described as negative (positive) energy solutions for λ^i_αs).

In the second case (QCD) the fundamental fields of the theory, the left-handed λ^i_αs and λ^{ci}_αs, transform as **3** and **3*** of $SU(3)$, respectively, and are related by charge conjugation. Positive energy solutions for λ^i_αs are identified with $-\frac{1}{2}$ helicity states of massless quarks and positive energy solutions for λ^{ci}_αs describe $-\frac{1}{2}$ helicity states of massless antiquarks. Negative energy solutions for λ^i_αs and λ^{ci}_αs describe, respectively, antiquarks and quarks with helicity $+\frac{1}{2}$. Since the representation R is real, mass terms are possible.

For a vector-like theory, like QCD, with charge conjugate pairs of Weyl spinors λ and λ^c, both transforming as $(\frac{1}{2}, 0)$ of the Lorentz group but as R and $R^\star$ representations of an internal symmetry group, it is convenient to use the Dirac four-component spinors (in the presence of the mass term in (1.68) λ and $\bar{\lambda}^c$ can be interpreted as describing opposite helicity (chirality) states of the same massless particle which mix with each other due to the mass term)

$$\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{c\dot{\alpha}} \end{pmatrix} \qquad \text{or} \qquad \Psi^c = \begin{pmatrix} \lambda^c_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{1.70}$$

transforming as R and $R^\star$, respectively, under the group of internal symmetries. The lagrangian (1.68) can be rewritten as

$$\mathcal{L} = \mathrm{i}\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \tag{1.71}$$

where we have introduced $\bar{\Psi} \equiv (\lambda^{c\alpha}, \bar{\lambda}_{\dot{\alpha}})$ to recover the mass term of (1.68). Writing $\bar{\Psi} = \Psi^\dagger \gamma^0$ and comparing the kinetic terms of (1.68) and (1.71) we define the matrices γ^μ. They are called Dirac matrices in the chiral representation and are given in Appendix A. Also in Appendix A we derive the Lorentz transformations for the Dirac spinors. Another possible hermitean mass term, $-\mathrm{i}m\bar{\Psi}\gamma_5\Psi$, can be rotated away by the appropriate phase transformation of the spinors λ and λ^c.

Four-component notation can also be introduced and, in fact, is quite convenient for Weyl spinors λ_α and $\bar{\chi}^{\dot{\alpha}}$ transforming as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ under the Lorentz group. We define four-component chiral fermion fields as follows

$$\Psi_\mathrm{L} = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \qquad \Psi_\mathrm{R} = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{1.72}$$

They satisfy the equations

$$\left.\begin{aligned} \Psi_\mathrm{L} &= \frac{1-\gamma_5}{2}\Psi_\mathrm{L} \equiv P_\mathrm{L}\Psi_\mathrm{L}, \qquad P_\mathrm{R}\Psi_\mathrm{L} = 0 \\ \Psi_\mathrm{R} &= \frac{1+\gamma_5}{2}\Psi_\mathrm{R} \equiv P_\mathrm{R}\Psi_\mathrm{R}, \qquad P_\mathrm{L}\Psi_\mathrm{R} = 0 \end{aligned}\right\} \tag{1.73}$$

which define the matrix γ_5 in the chiral representation. By analogy with the nomenclature introduced for λ_α and $\bar{\chi}^{\dot{\alpha}}$ the fields Ψ_L and Ψ_R are called left- and right-handed chiral fields, respectively. Possible Lorentz invariants are of the form $\bar{\Psi}_\mathrm{R}\Psi_\mathrm{L} \equiv \chi^\alpha\lambda_\alpha$ and $\bar{\Psi}_\mathrm{L}\Psi_\mathrm{R} \equiv \bar{\lambda}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$. The invariants $\lambda^\alpha\lambda_\alpha$ and $\bar{\lambda}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}$ can be written as $\Psi_\mathrm{L}^T C\Psi_\mathrm{L}$ and $\bar{\Psi}_\mathrm{L} C\bar{\Psi}_\mathrm{L}^T$, respectively, with the matrix C defined in (A.104). In particular, for a set of Weyl spinors λ_α and their complex conjugate

$\bar{\lambda}^{\dot{\alpha}}$, transforming as R and $R^\star$, respectively, under the internal symmetry group, the sets†

$$\Psi_{\rm L}^i = \begin{pmatrix} \lambda_\alpha^i \\ 0 \end{pmatrix} \qquad \Psi_{\rm R}^{ci} = \begin{pmatrix} 0 \\ \bar{\lambda}^{i\dot{\alpha}} \end{pmatrix} \tag{1.74}$$

also transform as R and $R^\star$, respectively. For instance, in ordinary QED if $\Psi_{\rm L}$ has a charge -1 $(+1)$ under the $U(1)$ symmetry group, we can identify it with the left-handed chiral electron (positron) field. The $\Psi_{\rm R}^c$ is then the chiral right-handed positron (electron) field.

Lorentz invariants $\overline{\Psi_{\rm L}^i}\Psi_{\rm R}^{cj}$ and $\overline{\Psi_{\rm R}^{ci}}\Psi_{\rm L}^j$ (or a combination of them) are generically not invariant under the internal symmetry group but such invariants can be constructed by coupling them, for example, to scalar fields transforming properly under this group (see later). The kinetic part of the lagrangian for the chiral fields is the same as in (1.71) but no mass term is allowed.

We note that $\gamma^5 = \gamma_5 = {\rm i}\gamma^0\gamma^1\gamma^2\gamma^3$ and that the following algebra is satisfied:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{1.75}$$

Finally, for objects represented by a Weyl field λ_α with no internal charges or transforming as a real representation of an internal symmetry group we can introduce the so-called Majorana spinor

$$\Psi_{\rm M} = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} \tag{1.76}$$

In this notation the lagrangian given by (1.65) and (1.66) reads

$$\mathcal{L}_{\rm M} = \frac{\rm i}{2}\bar{\Psi}_{\rm M}\gamma^\mu\partial_\mu\Psi_{\rm M} - \frac{m}{2}\bar{\Psi}_{\rm M}\Psi_{\rm M} \tag{1.77}$$

The equations of motion for free four-dimensional spinors have the form

$$\left({\rm i}\gamma^\mu\partial_\mu - m\right)\Psi = 0 \tag{1.78}$$

for the Dirac spinors and with $m = 0$ for the chiral spinors. In the latter case it is supplemented by the conditions (1.73). The solutions to the equations of motion for the Weyl spinors and for the four-component spinors in the chiral and Dirac representations for the γ^μ matrices are given in Appendix A.

† The notation in (1.74) has been introduced by analogy with (1.70). Note, however, that the superscript 'c' in the spinors $\Psi_{\rm R}^{ci}$ is now used even though λ_α and $\bar{\lambda}^{\dot{\alpha}}$ are not charge conjugate to each other (since they are in different Lorentz group representations). Thus, unlike the spinors in (1.70) the spinors in (1.74) are defined for both chiral and vector-like spectra of the fundamental fields. For a vector-like spectrum we have $\Psi_{\rm R}^{ci} = (\Psi^{ci})_{\rm R}$.

A few more remarks on the free Dirac spinors should be added here. The momentum conjugate to Ψ is

$$\Pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \Psi_\alpha)} = \mathrm{i}\Psi_\alpha^\dagger \tag{1.79}$$

For the hamiltonian density we get

$$\mathcal{H} = \Pi \partial_0 \Psi - \mathcal{L} = \Psi^\dagger \gamma^0 (-\mathrm{i}\boldsymbol{\gamma} \cdot \boldsymbol{\partial} + m)\Psi \tag{1.80}$$

and the energy–momentum tensor density reads

$$\Theta^{\mu\nu} = \mathrm{i}\bar{\Psi}\gamma^\mu \frac{\partial}{\partial x_\nu}\Psi \tag{1.81}$$

((1.81) is obtained after using the equations of motion). The angular momentum density

$$M^{\kappa,\mu\nu} = \mathrm{i}\bar{\Psi}\gamma^\kappa \left(x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu} - \frac{\mathrm{i}}{2}\sigma^{\mu\nu} \right)\Psi \tag{1.82}$$

(where four-dimensional $\sigma_{\mu\nu} = (\mathrm{i}/2)[\gamma_\mu, \gamma_\nu]$) gives the conserved angular momentum tensor

$$M^{\mu\nu} = \int \mathrm{d}^3 x \, M^{0\mu\nu} \tag{1.83}$$

The lagrangian (1.65) can be extended to include various interaction terms. For instance, for a neutral (with respect to the internal quantum numbers) Weyl spinor λ interacting with a complex scalar field ϕ (also neutral) the dimension four ($[m]^4$ in units $\hbar = c = 1$) Yukawa interaction terms are:

$$\lambda\lambda\phi, \qquad \lambda\lambda\phi^\star, \qquad \bar{\lambda}\bar{\lambda}\phi^\star, \qquad \bar{\lambda}\bar{\lambda}\phi \tag{1.84}$$

Similar terms can be written down for several spinor fields $(\lambda, \chi, \ldots)$ interacting with scalar fields. Additional symmetries, if assumed, would further constrain the allowed terms. These may be internal symmetries or, for instance, supersymmetry (see Chapter 15). The simplest supersymmetric theory, the so-called Wess–Zumino model, describing interactions of one complex scalar field $\phi(x)$ with one chiral fermion $\lambda(x)$ has the following lagrangian†

$$\begin{aligned} \mathcal{L}_{\mathrm{WZ}} = {} & \mathrm{i}\bar{\lambda}\bar{\sigma} \cdot \partial\lambda - \tfrac{1}{2}m(\lambda\lambda + \bar{\lambda}\bar{\lambda}) + \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi \\ & - g(\phi\lambda\lambda + \phi^\star\bar{\lambda}\bar{\lambda}) - gm\phi^\star\phi(\phi^\star + \phi) - g^2(\phi^\star\phi)^2 \end{aligned} \tag{1.85}$$

The absence of otherwise perfectly allowed coupling $-g'(\phi^\star\lambda\lambda + \phi\bar{\lambda}\bar{\lambda})$ is due to supersymmetry. In the four-component notation we get $(\phi\lambda\lambda + \phi^\star\bar{\lambda}\bar{\lambda}) = \bar{\Psi}_{\mathrm{M}} P_{\mathrm{L}} \Psi_{\mathrm{M}}\phi + \bar{\Psi}_{\mathrm{M}} P_{\mathrm{R}} \Psi_{\mathrm{M}}\phi^\star$, where Ψ_{M} is given by (1.76).

† This form of the lagrangian follows after eliminating auxiliary fields via their (algebraic) equations of motion, see Chapter 15.

In general, in the four-component notation the Yukawa terms have the generic structure $(\bar{\Psi}^i_{\rm R}\Psi^j_{\rm L} \pm \bar{\Psi}^j_{\rm L}\Psi^i_{\rm R})\Phi^k(\Phi^{k\star})$ where the chiral fields are defined by (1.72) and the internal symmetry indices are properly contracted.

If λ^is transform as a representation R (with hermitean generators T^a) of an internal symmetry group, the interactions with vector fields A^a_μ in the adjoint representation of the group (as for the gauge fields, see Section 1.3) take the form:

$$\bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu = \overline{\Psi^i_{\rm L}}\gamma^\mu(T^a)_i{}^j\Psi_{{\rm L}j}A^a_\mu$$

with $\Psi_{\rm L}$ given by (1.72).

For a vector-like theory, the lagrangian (1.68) in which λ^{ci} transforms as a representation R^* of the group (with generators $-T^{a*}$, see Appendix E) can be supplemented by terms like:

$$\begin{aligned}&\bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu + \bar{\lambda}^c_i\bar{\sigma}^\mu(-T^{a*})^i{}_j\lambda^{cj}A^a_\mu\\ &\quad = \bar{\lambda}^i\bar{\sigma}^\mu(T^a)_i{}^j\lambda_j A^a_\mu + \lambda^{cj}\sigma^\mu(T^a)_j{}^i\bar{\lambda}^c_j A^a_\mu\\ &\quad = \bar{\Psi}^i\gamma^\mu(T^a)_i{}^j\Psi_j A^a_\mu\end{aligned}$$

In the last form (in which $\Psi(x)$ is given by (1.70)) this interaction can be added to the lagrangian (1.71). For the abelian field A_μ this is the same as the minimal coupling of the electron to the electromagnetic field ($p_\mu \to p_\mu + eA_\mu$). Other interaction terms of a Dirac field (1.70), for example, with a scalar field $\Psi(x)$ or a vector field $A_\mu(x)$, are possible as well. A Lorentz-invariant and hermitean interaction part of the lagrangian density may consist, for example, of the operators:

$$\bar{\Psi}\Psi\Phi, \qquad \bar{\Psi}\gamma^5\Psi\Phi, \qquad \bar{\Psi}\gamma^5\gamma_\mu A^\mu\Psi \tag{1.86}$$

$$\frac{1}{M}\bar{\Psi}\Psi\Phi\Phi, \qquad \frac{1}{M}\bar{\Psi}\gamma_\mu A^\mu\Psi\Phi, \qquad \frac{1}{M}\bar{\Phi}\Phi A_\mu A^\mu, \qquad \frac{1}{M^2}\bar{\Psi}\Psi\bar{\Psi}\Psi \qquad \text{etc.} \tag{1.87}$$

The first two terms are Yukawa couplings (scalar and pseudoscalar). The terms in (1.87) require dimensionful coupling constants, with the mass scale denoted by M. The last term, describing the self-interactions of a Dirac field, is called the Fermi interaction. It is straightforward to include the interaction terms in the equation of motion.

1.3 Gauge field theories

$U(1)$ *gauge symmetry*

As a well-known example, we consider gauge invariance in electrodynamics. It is often introduced as follows: consider a free-field theory of n Dirac particles with the lagrangian density

$$\mathcal{L} = \sum_{i=1}^{n} \left(\bar{\Psi}_i \mathrm{i}\gamma^\mu \partial_\mu \Psi_i - m \bar{\Psi}_i \Psi_i \right) \tag{1.88}$$

Then define a $U(1)$ group of transformations on the fields by

$$\Psi_i'(x) = \exp(-\mathrm{i}q_i \Theta) \Psi_i(x) \tag{1.89}$$

where the parameter q_i is an eigenvalue of the generator Q of $U(1)$ and numbers the representation to which the field Ψ_i belongs. The lagrangian (1.88) is invariant under that group of transformations.

By Noether's theorem (see Section 1.1), the $U(1)$ symmetry of the lagrangian (1.88) implies the existence of the conserved current

$$j_\mu(x) = \sum_i q_i \bar{\Psi}_i \gamma_\mu \Psi_i \tag{1.90}$$

and therefore conservation of the corresponding charge (1.17). We now consider gauge transformations (local phase transformations in which Θ is allowed to vary with x)

$$\Psi_i'(x) = \exp[-\mathrm{i}q_i \Theta(x)] \Psi_i(x) \tag{1.91}$$

It is straightforward to verify that lagrangian (1.88) is not invariant under gauge transformations because transformation of the derivatives of fields gives extra terms proportional to $\partial_\mu \Theta(x)$. To make the lagrangian invariant one must introduce a new term which can compensate for the extra terms. Equivalently, one should find a modified derivative $D_\mu \Psi_i(x)$ which transforms like $\Psi_i(x)$

$$[D_\mu \Psi_i(x)]' = \exp[-\mathrm{i}q_i \Theta(x)] D_\mu \Psi_i(x) \tag{1.92}$$

and replace ∂_μ by D_μ in the lagrangian (1.88).

The derivative D_μ is called a covariant derivative. The covariant derivative is constructed by introducing a vector (gauge) field $A_\mu(x)$ and defining†

$$D_\mu \Psi_i(x) = [\partial_\mu + \mathrm{i}q_i e A_\mu(x)] \Psi_i(x) \tag{1.93}$$

where e is an arbitrary positive constant. The transformation rule (1.92) is ensured if the gauge field $A_\mu(x)$ transforms as:

$$A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \Theta(x) \tag{1.94}$$

Covariant derivatives play an important role in gauge theories. In particular

† The sign convention in the covariant derivative is consistent with the standard minimal coupling in the classical limit (Bjorken and Drell (1964); note that $e < 0$ in this reference).