An Introduction to Economic Dynamics

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Chapter 1
Introduction

In this chapter we shall introduce some basic concepts of dynamics. In order to illustrate these we shall consider just one example. This example is a simple linear model. Why such a linear equation explains what is of interest we shall not consider here. Our main aim is to define and elaborate on dynamic concepts that we shall use throughout this book. Our second aim is to show how such a model can be set up on a spreadsheet and then investigated interactively.

1.1 Definitions and concepts

Dynamics is concerned with how things change over time. The passage of time is a crucial element in any dynamic process. Whether a variable remains the same value at two points in time or whether it is different is not the essential issue, what matters is that time has elapsed between the value of the variable on the first occasion and the value it takes on the second. Time cannot be reversed.

In dynamics we must specify the point in time for any given variable. If we are concerned about national income, price or profits we need to specify the level of income at a point in time, the price at a point in time and profits at some point in time. As time moves on the value these variables take may change. We will specify time in this book by referring to periods: period 0, usually referring to the initial point in time, then period 1, period 2, and so on. Consequently we shall denote this $t=0$, 1, 2, etc. If our variable of interest is price, say, which we denote as p, then $p(0)$ refers to the price at time period 0, the present, $p(1)$ the price at time period 1, $p(2)$ the price at time period 2, and so on. Unfortunately, referring to price in this way allows us to refer only to *future* prices. Sometimes we wish to talk about what the price was in the previous period, or the price two periods ago. In order to do this we sometimes say that $p(t)$ is the price in period t, the price now, $p(t+1)$ the price in the next period, and $p(t+2)$ the price two periods from now. Doing this then allows us to refer to price in the previous period, $p(t-1)$, and the price two periods in the past, $p(t-2)$, etc. Which we use depends on what we are discussing, but the context should make it quite clear. If our model is continuous, then $p(t)$ is a continuous function of time. There is a price for each instant of time. We consider continuous models in section 1.9.

How a variable changes over time depends on what determines that variable. What determines a variable is formulated by means of models. In other words, a model is an explanation of how the variable comes about: how it takes on the value that it does, how it is related to other variables, and how it changes over time. A model that refers to no passage of time is called a static model. Elementary economics has many static models. The model of demand and supply, which determines the equilibrium price, is a typical static model. Equilibrium price is determined by the equality between what is demanded and what is supplied: where the demand curve intersects the supply curve. If demand rises, and the demand curve shifts to the right, then equilibrium price will rise. When we compare one equilibrium with another we are concerned with comparative statics. We are simply comparing the two or more equilibrium points. How the variable got to the new equilibrium is not really considered. To do this would require some dynamic process to be specified. Usually in the theory of demand and supply the movement is assumed to be instantaneous. Or, put another way, that adjustment all happens in the *same* time period, so that it is unnecessary to specify time. On the other hand, if we wish to specify the time path of a variable between one equilibrium point and another, then we must set out a dynamic model which explicitly explains the movement of the variable over time. In other words, a dynamic model must involve time explicitly.

Notice here that the model comes from the subject. It comes from our understanding of how the world works. The world is a complex place and we simplify by forming a model. The model sets out the relationships between the crucial elements of the system we are interested in. Models involve abstractions and simplifications. An economic model will concentrate on the economic aspects of a system while a sociological model would concentrate on the social aspects of the same system. In this book we are concerned only with economic models. The subject matter of economics is usually divided into microeconomics and macroeconomics. Microeconomics is concerned with individual units, such as choices made by individuals, profits made by firms, decisions about supplying labour at different wage rates, and so on. Macroeconomics is concerned with aggregate variables at the economy level such as unemployment, national income and the general price level. A large part of studying economics is coming to an understanding of microeconomic models and macroeconomic models. In elementary courses in economics these models are usually static models. Time does not enter them explicitly and attention is usually directed towards the determination of equilibrium conditions.

An equilibrium of a model is where the system settles down and, once there, there is no reason for the system to move. It is often thought of in mechanical terms as a balance of forces. In demand and supply, for example, demand represents one force and supply another. When demand equals supply then the forces are in balance and the system is in equilibrium. The price that establishes this balance of force is then referred to as the equilibrium price. Much attention in economics is paid to what determines the equilibrium of a model and how that equilibrium changes when some feature of the system changes. But most elementary textbooks stop at this point. But consider for a moment. To establish that a system has an equilibrium just establishes whether an equilibrium exists or not. It cannot guarantee that the system will ever achieve that equilibrium. When attention is directed at the attainment or not of the equilibrium we are dealing with its stability or instability. We refer to this simply as the condition of stability of the equilibrium. But to consider the stability of an equilibrium we need to know what happens to the variable over time. If a variable over time tends towards the equilibrium value, then we say it is stable. If a variable moves away from the equilibrium value then we say it is unstable. (We shall explain this more formally later in the book.) Notice that it is the stability of the equilibrium which we are referring to. Furthermore, any discussion of stability must involve the passage of time explicitly, and so stability is a dynamic consideration of the model. To illustrate this in simple terms take a bowl and (gently) drop an egg down the side. The egg will slip down the side, rise up the other, and steadily come to rest at the bottom of the bowl. The movements around the base get smaller and smaller over time. The base of the bowl represents a stable equilibrium. We know it is an equilibrium because the egg stops moving, and will remain there until it is disturbed. Furthermore, if gently moved a little from the base, it will soon return there. Now place the egg carefully on a bowl placed upside down. If placed carefully, then the egg will remain in that position. It is equilibrium. There is a balance of forces. But move the egg just a little and it will topple down the side of the bowl. It does not matter which direction it is moved, once moved the egg will move away from the top of the bowl. In other words, the top of the bowl is an unstable equilibrium. In this book we shall be considering in some detail the stability of equilibrium points. In this example the movement of the egg was either towards the equilibrium or away from it. But in some systems we shall be considering it is possible for a variable to move *around* the equilibrium, neither moving towards it nor away from it! Such systems exhibit oscillatory behaviour.

Here we have introduced the reader to only some of the concepts that we will be dealing with. It will be necessary to formalise them more carefully. We shall do this in terms of the economic models we shall consider.

1.2 Dynamic models

Consider the following equation, which we shall assume for the moment comes from some theory of economics explaining the variable x.

$$x(t+1) = 3 + \tfrac{1}{2}x(t) \tag{1.1}$$

Since the variable x at time $t+1$ is related to the same variable in the previous period we call such models recursive. This is true even if more than one time period in the past is involved in the relationship. This recursive model is also linear, since the equation itself is a linear equation. In more complex models nonlinear equations can arise, but they are still recursive if they are related to the same variable in early periods. If the relationship is for just one previous period, then we have a first-order recursive equation; if it is for two periods, then we have a second-order recursive equation, and so on.

Now in itself this is not sufficient to specify the time path of the variable x.

We need to know its *starting value*. For the moment let this be $x(0)=10$. Obviously, if this is the case then $x(1)=3+\frac{1}{2}(10)=8$ and $x(2)=3+\frac{1}{2}(8)=7$. The sequence of $x(t)$ generated over time is then 10, 8, 7, 6.5, 6.25 … We can learn quite a bit from this equation. First, the change in the sequence is getting smaller and appears to be getting close to some number. If the series was extended for many more periods it would indicate that the series is getting closer and closer to the number 6. Is this a coincidence? No, it is not. The number 6 is the equilibrium of this system. Can we establish this? Yes, we can. If the system is in equilibrium it is at rest and so the value the variable x takes in each period is the same. Let us call this x^*. Then it follows that $x(t-1)=x(t)=x^*$, and so $x^*=3+\frac{1}{2}x^*$ or $x^*=6$. Mathematicians often call equilibrium points fixed points and we shall use the two terms interchangeably. But we can say much more. From the solution we have just derived it is clear that there is only *one* fixed point to this system: one equilibrium.

It is very useful to display first-order recursive systems of this type on a diagram that highlights many features we shall be discussing. On the horizontal axis we measure $x(t)$ and on the vertical axis we measure $x(t+1)$. Next we draw a 45°-line. Along such a line we have the condition that $x(t+1)=x(t)$. This means that any such equilibrium point, any fixed point of the system, must lie somewhere on this line. Next we draw the equation $3+\frac{1}{2}x(t)$. This is just a straight line with intercept 3 and slope $\frac{1}{2}$. For this exercise we assume a continuous relationship. The situation is shown in figure 1.1. It is quite clear from this figure that the line $3+\frac{1}{2}x(t)$ cuts the 45°-line at the value 6, which satisfies the condition

$$x(t+1)=x(t)=x^*=6$$

It is also quite clear from figure 1.1 that this line can cut the 45°-line in only one place. This means that the equilibrium point, the fixed point of the system, is unique.

Given the starting value of $x(0)=10$, the next value is found from a point on the line, namely $x(1)=3+\frac{1}{2}x(0)=3+\frac{1}{2}(10)=8$. At this stage the value of $x(1)$ is read on the vertical axis. But if we move horizontally across to the 45°-line, then we can establish this *same* value on the horizontal axis. Given this value of $x(1)$ on the horizontal axis, then $x(2)$ is simply read off from the equation once again, namely $x(2)=3+\frac{1}{2}x(1)=3+\frac{1}{2}(8)=7$. Continuing to perform this operation will take the system to the equilibrium point $x^*=6$. The line pattern that emerges is referred to as a cobweb. We shall consider these in more detail in chapter 2.

It would appear on the face of it that the fixed point $x^*=6$ is a stable fixed point, in the sense that the sequence starting at $x(0)=10$ converges on it. But we must establish that this is true for other starting values. This may have been an exception! Suppose we take a starting value below the equilibrium point, say $x(0)=3$. If we do this, the sequence that arises is 3, 4.5, 5.25, 5.625 … So once again we note that the sequence appears to be converging on the fixed point of the system. This is also shown in figure 1.1. It is very easy to establish that no matter what the starting value for the variable x, the system will over time converge on the fixed point $x^*=6$. Not only is this fixed point stable, but

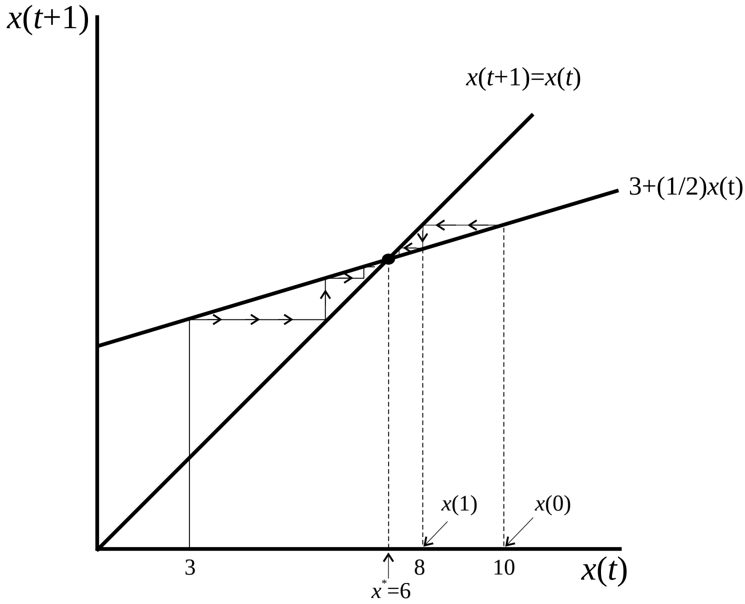


Figure 1.1

also it is said to be globally stable. The word ‘global’ indicates that it does not matter what value of x is taken as a starting value, whether near to the fixed point or far away from the fixed point, the system will always converge on the fixed point.

To reiterate, we have established that the system has an equilibrium point (a fixed point), that there is only one such equilibrium point and that this fixed point is globally stable. This is a lot of information.

1.3 Deterministic dynamical models

We can use the model we have just outlined to clarify more clearly what we mean by a ‘dynamic model’. In doing this, however, let us generalise the initial point. Let this be denoted $x(0) = x_0$, then the system can be written

$$x(t+1) = 3 + \tfrac{1}{2}x(t) \quad x(0) = x_0 \tag{1.2}$$

This is a deterministic dynamical model (or deterministic dynamical system). It is a dynamic system because it deals with the value of the variable x over time. Given $x(0) = x_0$, then we can trace out the whole series of value of $x(t)$, for all time periods t from period 0 onwards. Notice that the series is crucially dependent on the initial condition. A different initial condition, as in our example above, will lead to quite a different series of numbers, although they will in this instance converge on the same fixed point. Why have we referred to it as ‘deterministic’? It is deterministic because given the *same* initial value, the sequence of numbers is always the *same*. The initial condition and the specification of the recursive equation determine the sequence. There is no random element entering the series. Even if we calculate the sequence on a computer the numbers will be identical for the same starting value. It does not matter which software we use or which chip is contained in the computer. The whole system is deterministic.

Let us generalise the model. Suppose

$$x(t+1) = a + bx(t) \quad x(0) = x_0 \tag{1.3}$$

This is still a deterministic dynamical system. However, to establish the sequence of $x(t)$ over time we need to know the values of a and b, which are referred to as the parameters of the system. Parameters are constants of the system and typically capture the structure of the problem under investigation. They are therefore sometimes called structural parameters. We now have the three ingredients that are necessary to specify a deterministic dynamical system. They are:

(1) the initial condition, namely $x(0) = x_0$
(2) the values of the parameters, here the values of a and b
(3) the sequence of values over time of the variable x.

As we shall see later, the fact that the system is deterministic does not mean that it may not appear like a random series. It simply means that given the initial condition and the same values for the parameters, then the sequence of values that are generated will always be the same no matter what they look like.

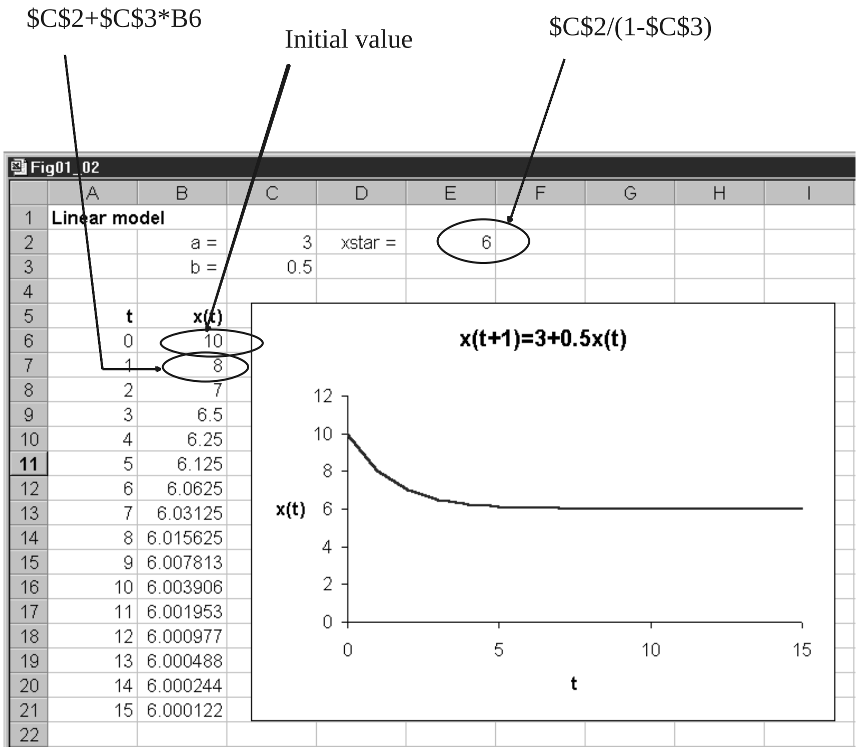
1.4 Dynamical systems on a spreadsheet

We shall frequently be displaying dynamical systems on a spreadsheet and so we shall use our present model to illustrate how this is done. Spreadsheets are ideal mediums for investigating recursive systems, and a great deal of dynamic investigation can easily be undertaken with their help. Using spreadsheets avoids the necessity of establishing complex formulas for solution paths. Of course the more one understands about such solution methods, the more one can appreciate the nature of the dynamic system under investigation. For individuals wishing to know such solution methods they will find these in my *Economic Dynamics* (Shone, 1997).

From the very outset we want to set up the model in general terms so we can undertake some analysis. This may involve changing the initial value and/or changing the value of one or more of the parameters. The situation is shown in figure 1.2. At the top of the spreadsheet we have the values of the two parameters a and b. The values themselves are in cells C2 and C3, respectively. When using spreadsheets it is essential to understand from the very outset that cells can have absolute addresses or relative addresses. An absolute address is distinguished from a relative address by having the $-sign precede the row and column designation: C3 is a relative address while C3 is an absolute address. The importance of this distinction will become clear in a moment.

In the first column we place our time periods, $t = 0, 1, 2$, etc. It is not necessary to type in these values, and it would be tedious to do so if you wanted to investigate the dynamics of a model over 500 time periods or even 2000! Most

Figure 1.2



	A	B	C	D	E
1	Linear model				
2		a =	3	xstar =	6
3		b =	0.5		
4					
5	t	x(t)			
6	0	10			
7	1	8			
8	2	7			
9	3	6.5			
10	4	6.25			
11	5	6.125			
12	6	6.0625			
13	7	6.03125			
14	8	6.015625			
15	9	6.007813			
16	10	6.003906			
17	11	6.001953			
18	12	6.000977			
19	13	6.000488			
20	14	6.000244			
21	15	6.000122			
22					

software packages have a ‘Fill’ command. You simply specify the initial value (here 0) and then block down and request a fill with the incremental value included, here an increment of 1. That is all that is necessary. It is useful to include the time periods because it then becomes easier to graph the series $x(t)$. Since the row headings take place along row 5, the time periods are in cells A6, A7, etc. Next we place in cell B6 the initial value. In this example we insert the value 10.

At this stage it is essential to distinguish the absolute and relative address. Since the parameter values will always be the same, we need to refer to the absolute value in cell C2 and C3, i.e. the parameter a has the value in cell C2 while the parameter b has the value in cell C3. We now need to write a formula in cell B7. A comparison between the algebraic formula and the spreadsheet formula is useful here. These are for the value $x(1)$

$$= a + bx(0)$$
$$= \$C\$2 + \$C\$3{*}B6$$

Notice that B6 is a relative address, it simply refers to the previous value of x, which in this instance is the initial value 10. Also notice that it is necessary when specifying formulas in a spreadsheet to indicate a multiplication by the ‘star’ symbol. Once this formula is entered it is replaced by the value it takes, in this example the value 8.

The power of the spreadsheet really comes into play at this point. Consider for a moment what we would do if we wished to calculate the value of $x(2)$.

This value is positioned in cell B8. Again comparing the algebraic specification and the spreadsheet will help clarify what is going on

$$\begin{aligned}&= a + bx(1)\\&= \$C\$2 + \$C\$3 * B7\end{aligned}$$

Because the parameter values have absolute addresses, their values do not change. However, B7 is a relative address and refers to the cell immediately above. That in turn has already been calculated. But the spreadsheet formula in cell B8 is almost identical to the formula in cell B7, the only difference is the value which x takes, which is always the value in the cell immediately above. If you copy the formula in cell B7 to the clipboard and paste it down for as many periods as you are considering, the computations are immediately carried out, with the value of x changing in the formula each time to be the value of x in the cell immediately above. This can be done because the cell involves a relative address (along with some absolute addresses) and this relative address keeps changing. Absolute addresses do not change. So you can paste down 13, 488 times or even 1,998 times with one click.

We have laboured this point here because it is the feature we shall be using throughout. It also indicates that when dealing with dynamic systems on a spreadsheet it is useful to set out the parameter values and then refer to their absolute addresses and ensure formulas are entered *and changed* to include the appropriate absolute and relative addresses. They need to be changed since all formulas are entered with only relative addresses.

The spreadsheet involves one other computed value, namely the fixed point of the system. Since

$$x^* = a + bx^*$$

then

$$x^* = \frac{a}{1-b} \tag{1.4}$$

In the spreadsheet we label the fixed point as 'xstar =' and its value is placed in cell E2 where this value is

$$= \$C\$2/(1 - \$C\$3)$$

Consequently any change to the parameter values is immediately reflected in a change to the equilibrium value.

> Note
> It is always useful to check that you have entered formulas in the main body of the computations. This can easily be accomplished. Copy the equilibrium value to the cell containing the value for $x(0)$, cell B6. If your formula is correctly entered then *every* entry in column B should be the same equilibrium value!

One final thing to do is to graph the series of $x(t)$ against time, t. This is simply a X-Y plot with time on the horizontal axis and the variable x on the vertical axis. Here we assume you are familiar with your spreadsheet's graphing facility. Typically spreadsheets allow you either to place a graph on its own sheet, or as a graphic item on the sheet where the calculations are being done. This latter position is very useful when you wish to experiment with your model because then you can see immediately the impact of changing some element of the model. Placing it on its own sheet is useful if you wish to have a printout of the graph. We shall experiment with the model in section 1.5. To insert the graph, block cells A6:B21 and invoke the chart wizard. Choose the X-Y plot and choose the option with the points joined. The wizard automatically knows that the first column (cells A6:A21) is the values on the x-axis. We have also included a title and labels for the two axes. We also have turned the y-axis label through 90°. Figure 1.2 shows the resulting time path of $x(t)$. In order to see the dynamics of the path more clearly, we have suppressed the points and joined the points up with a continuous line. The plot readily reveals the stability of the equilibrium, with the path of $x(t)$ starting at the value 10 and tending to the equilibrium value of 6.

1.5 Experimentation

It is now time to experiment with the model in order to investigate the characteristics of its dynamics. We shall leave this up to the reader, and here just indicate what you should expect to observe.

1.5.1 Changing initial conditions

We stated above that this model was globally stable; that no matter what the initial value was, the system would always converge on the equilibrium value, (1.4). Verify this. Try for example $x(0)=3$, 0, 7, -2 and 25. No matter what value is chosen, the system will always converge on the value 6. Of course, sometimes it takes a long time to do this. If the initial value were 100, for example, then it takes a much longer time to reach the equilibrium value than if the initial value were 10.

1.5.2 Changing the parameter a

Raising (lowering) the value of the parameter a raises (lowers) the equilibrium value. This readily follows from the formula for the equilibrium value, but it is readily verified on the spreadsheet. It is also apparent from figure 1.1. A rise in the parameter a is a rise in the intercept in the formula $a+bx(t)$, and this will intersect the 45°-line further up. A fall in the parameter a will do the opposite. Such a change alters only the equilibrium value, the value of the fixed point. It has no bearing on the stability properties of that fixed point. The system remains convergent. Verify these statements by changing the value of the parameter a and choose again the same initial values for the variable x.

1.5.3 Changing the parameter b

Retain the initial value of $x(0)=10$ but now let $b=1.5$. Not only does the equilibrium become negative, with value -6, but also the system diverges away from the equilibrium value. The variable $x(t)$ just grows and grows. Let $b=-\frac{1}{2}$. The equilibrium value falls from 6 to 2. Furthermore, the values that x take oscillate above and below this value, but converge on it. If $b=-1.5$, the system still oscillates, but the oscillations diverge away from the equilibrium value, which is now 1.2. Finally take $b=-1$. Equilibrium becomes 1.5 and the system oscillates either side of this value indefinitely, with values -7 and 10, and the system neither moves towards the equilibrium or away from it.

It is apparent from these experimentations that changing the value of the parameter b can have drastic consequences on the dynamics of this system, far more dramatic an impact than occurs when the parameter a is altered.

Carry out some more experimentation with changes in the value of the parameter b. What you should conclude is the following:

(1) A value of $0<b<1$ leads to the system converging on the equilibrium value.

(2) A value of $b=1$ leads to no fixed point. (What does this imply about the graph of $x(t+1)$ against $x(t)$?)

(3) A value of $-1<b<0$ leads to the system oscillating, but converging on the equilibrium value.

(4) A value of $b=-1$ leads to oscillations between two values, neither moving toward nor away from the equilibrium value.

(5) A value of $b<-1$ leads to oscillations which diverge further and further from the equilibrium value.

All these statements are true regardless of the initial value taken by the system (other than the equilibrium value).

What began as a very simple linear model has led to a whole diversity of dynamic behaviour. It clearly illustrates that simply demonstrating that a model has an equilibrium point is not sufficient. It is vital to establish whether the system will converge or not converge on this equilibrium. It is essential to investigate the dynamics of the model.

1.6 Difference equations

The recursive system we have been analysing, represented here as (1.5)

$$x(t+1)=3+\tfrac{1}{2}x(t) \tag{1.5}$$

can be expressed in a different way which is often very revealing about its dynamics. If we subtract from both sides the same value, then we have not changed the characteristics of the system at all. In particular, the equilibrium value is unchanged and the stability/instability of the system is unchanged. Suppose, then, that we subtract from both sides the value $x(t)$, then we have

$$x(t+1)-x(t)=\Delta x(t+1)=3+\tfrac{1}{2}x(t)-x(t)=3+\left(\tfrac{1}{2}-1\right)x(t)$$

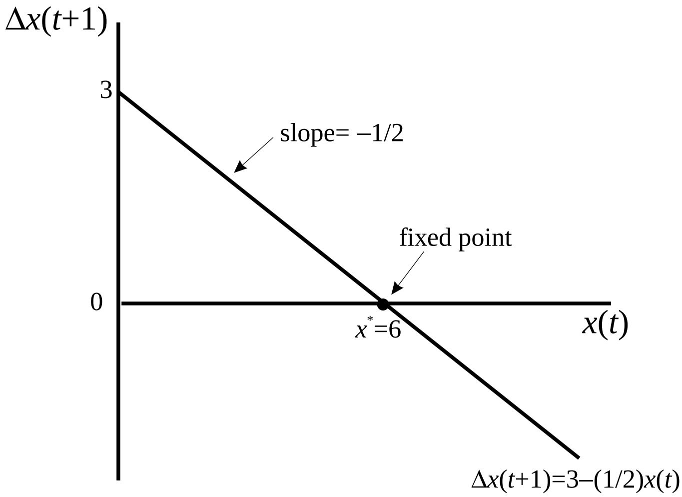


Figure 1.3

or

$$\Delta x(t+1) = 3 - \tfrac{1}{2}x(t) \tag{1.6}$$

This relationship is referred to as a **difference equation** because it expresses the difference $\Delta x(t+1) = x(t+1) - x(t)$ as a function of $x(t)$. It is also a **first-order difference equation** because we are considering only the first difference. The system is shown in figure 1.3, where we place $x(t)$ on the horizontal axis and $\Delta x(t+1)$ on the vertical axis. Of particular note is that the intercept is the value 3 and the slope of the line is $-\frac{1}{2}$.

Let us establish that the properties of the system are the same. Consider first the equilibrium value, the fixed point of the system. In equilibrium we know that $x(t+1) = x(t) = x^*$. Then it follows that $\Delta x(t+1) = x(t+1) - x(t) = 0$. Given this situation, then $0 = 3 - \frac{1}{2}x^*$ or $x^* = 6$. We have therefore verified that the equilibrium value is unchanged. In terms of figure 1.3, the equilibrium is where the equation $3 - \frac{1}{2}x(t)$ cuts the horizontal axis, because at this point $\Delta x(t+1) = 0$.

Now consider the stability or otherwise of the equilibrium point. Take the typical initial value we have been using of $x(0) = 10$ This value lies above the equilibrium value of 6, and so $\Delta x(t+1)$ is negative. If $\Delta x(t+1) < 0$ then $x(t+1) < x(t)$ and so $x(t)$ is falling over time. In fact this will continue to be so until the fixed point is reached. If, on the other hand, we take $x(0) = 3$, then $\Delta x(t+1) > 0$ and so $x(t+1) > x(t)$, and hence $x(t)$ is rising over time. Again, this will continue to be so until the fixed point is reached. Once again, therefore, we have demonstrated that the fixed point is 6 and that it is stable. Even more, no matter what value of $x(0)$ we take, the system will converge on the equilibrium. The equilibrium is unique and globally stable. The characteristic to take note of here is that the line that passes through the equilibrium in figure 1.3 is *negatively sloped* and cuts the x-axis at only one point.

Consider next the situation where $b = 1.5$. In this case

$$x(t+1) - x(t) = \Delta x(t+1) = 3 + 1.5x(t) - x(t) = 3 + \big(1.5 - 1\big)x(t)$$

or

(1.7) $$\Delta x(t+1) = 3 + 0.5x(t)$$

Is the equilibrium unchanged? No, it changes since

$$0 = 3 + 0.5x^*$$
$$x^* = -6$$

Also the line $3 + 0.5x(t)$ is *positively sloped.* At $x(0) = 10$ $\Delta x(t+1) > 0$ and so $x(t)$ is rising. The system is moving further away (in the positive direction) from the equilibrium value. A value of $x(t)$ less than -6 will readily reveal that $\Delta x(t+1) < 0$ and so $x(t)$ is falling, and the system moves further away (in the negative direction) from the equilibrium point. A linear system with a positively sloped difference equation, therefore, exhibits an unstable fixed point.

To summarise, for *linear* difference equations of the first order, if the difference equation has a nonzero slope, then a unique fixed point exists where the difference equation cuts the horizontal axis. If the difference equation is *negatively* sloped, then the fixed point of the system is unique and globally stable. If the linear difference equation is *positively* sloped, then the fixed point of the system is unique and globally unstable. We have demonstrated all this in previous sections. If $b = 1$ the slope is zero and no fixed point is defined. All we have done here is to show the same characteristics in a different way. It may not at this point seem obvious why we would do this. It is worth doing only if it gives some additional insight. It gives some, but admittedly not very much. Why we have laboured this approach, however, is that when we turn to two variables, it is much more revealing. We shall see this in later chapters.

1.7 Attractors and repellors

We noted that in our example if $-1 < b < 1$ then the system is stable and the sequence of points converges on the fixed point. It converges either directly if b is positive or in smaller and smaller oscillations if b is negative. If a trajectory (a sequence of points) approaches the fixed point as time increases, then the fixed point is said to be an attractor. On the other hand, if the sequence of points moves away from the fixed point, then the fixed point is said to be a repellor.

We can illustrate these concepts by means of the phase line. In constructing the phase line we make use of the difference equation representation of our recursive model. Our model is

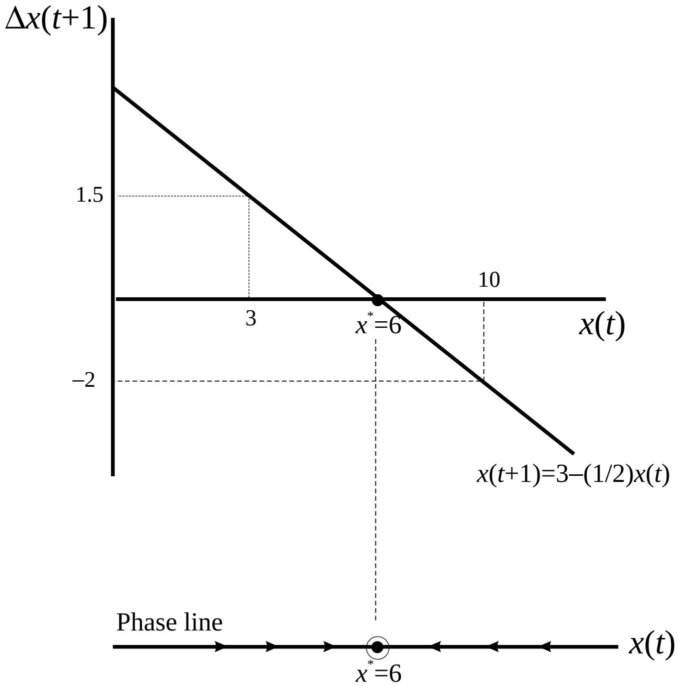
(1.8) $$x(t+1) = 3 + \tfrac{1}{2}x(t) \quad x(0) = 10$$

and the difference equation version of it is

(1.9) $$\Delta x(t+1) = 3 - \tfrac{1}{2}x(t) \quad x(0) = 10$$

This is shown in the upper part of figure 1.4. The fixed point, denoted x^*, is where the line $3 - \frac{1}{2}x(t)$ cuts the horizontal axis, which is at the value 6. The phase line simply denotes the variable $x(t)$, and on this line is marked any fixed

Figure 1.4



points (here we have only one), and arrows indicating the path of $x(t)$ over time. To the left of x^*, $\Delta x(t)$ is positive, and so $x(t)$ is rising over time. The arrows are therefore shown pointing to the right. Similarly, when the initial point is to the right of x^*, $\Delta x(t)$ is negative and so $x(t)$ is falling over time. The arrows are therefore shown pointing to the left. The phase line thus illustrates that the fixed point is attracting the system from any position on either side. We have already established that this is the only fixed point and that it is globally stable. Hence, for any initial value not equal to the equilibrium, the system will be attracted to the fixed point.

Consider next the recursive model

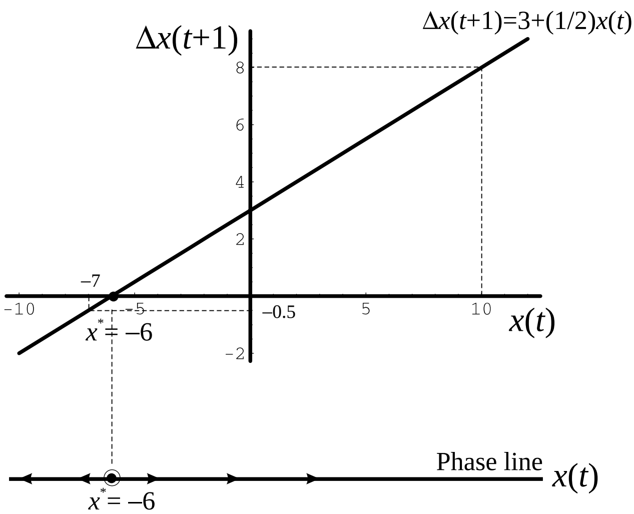
$$x(t+1) = 3 + 1.5x(t) \quad x(0) = 10 \tag{1.10}$$

with the difference equation version

$$\Delta x(t+1) = 3 + \tfrac{1}{2}x(t) \quad x(0) = 10 \tag{1.11}$$

The equilibrium point is $x^* = -6$, and the relationship $3 + \frac{1}{2}x(t)$ is positively sloped. The situation is shown in figure 1.5. Once again we place the phase line in the diagram below. For any initial point to the right of -6, then $\Delta x(t+1)$ is positive, and so $x(t)$ is rising over time. The arrows are therefore shown pointing to the right and the system is moving away from the fixed point. Similarly, when the initial point is to the left of x^*, $\Delta x(t)$ is negative and so $x(t)$ is falling over time. The arrows are therefore shown pointing to the left and once again the system is moving away from the fixed point. The phase line thus illustrates that the fixed point is repelling the system for any initial value not equal to the equilibrium value.

Figure 1.5



Fixed points that are attracting indicate stability while fixed points that repel indicate instability. But a fixed point can be neither of these, even in simple linear models. We noted this above when b was equal to minus unity. The system oscillated between two values: one above the equilibrium and one below the equilibrium. The system neither moved towards the fixed point nor away from it. In this case we observe a periodic cycle, and in this example the period is 2.[1]

1.8 Nonlinear dynamical systems

Although a considerable amount of analysis has taken place concerning linear models, it must always be kept in mind that in general the world is nonlinear, and it is necessary to model the topic of interest with nonlinear equations. Nonlinear models lead to far more diverse behaviour. They can lead to more than one equilibrium point, they can lead to a system exhibiting both stability or instability in different neighbourhoods and they can lead to cyclical behaviour of orders greater than two.

Our intention in this section is to present some introductory remarks about nonlinear systems and to introduce some new concepts. A fuller treatment will occur in later sections of this book. Although nonlinear systems are more complex and lead to more diverse behaviour, they can still be investigated in a fairly easy fashion with the aid of a spreadsheet.

Consider the following nonlinear recursive model

$$x(t+1) = c + ax^2(t) \quad x(0) = x_0 \tag{1.12}$$

As earlier, the equilibrium of the system is found by setting $x(t+1) = x(t) = x^*$, then

[1] Period cycles are explained more fully in chapter 10.

$$x^* = c + ax^{*2}$$
$$ax^{*2} - x^* + c = 0$$

with solutions

$$x_1^* = \frac{1 + \sqrt{1 - 4ac}}{2a} \text{ and } x_2^* = \frac{1 - \sqrt{1 - 4ac}}{2a} \qquad (1.13)$$

We immediately see from (1.13) that there are *two* fixed points to this system. Second, the fixed points are real valued only if $1 - 4ac \geq 0$. But if there are two fixed points to the system, then any consideration of stability or instability cannot be global; it must be in relation to a particular fixed point. When there is more than one fixed point we refer to local stability and local instability. The word 'local' indicates that we are considering stability only in a (small) neighbourhood of the fixed point.

Consider the following nonlinear recursive system

$$x(t+1) = 2 - \tfrac{1}{2}x^2(t) \quad x(0) = x_0 \qquad (1.14)$$

which leads to equilibrium points $x_1^* = -1 + \sqrt{5} = 1.23607$ and $x_2^* = -1 - \sqrt{5} = -3.23607$ (see box 1).

Box 1 Solving quadratic equations with a spreadsheet

We shall be solving quadratic equations frequently in this book and so it will be useful to set the solutions up on a spreadsheet. Let any quadratic equation be written in the form

$$ax^2 + bx + c = 0$$

then we know that the solutions are given by

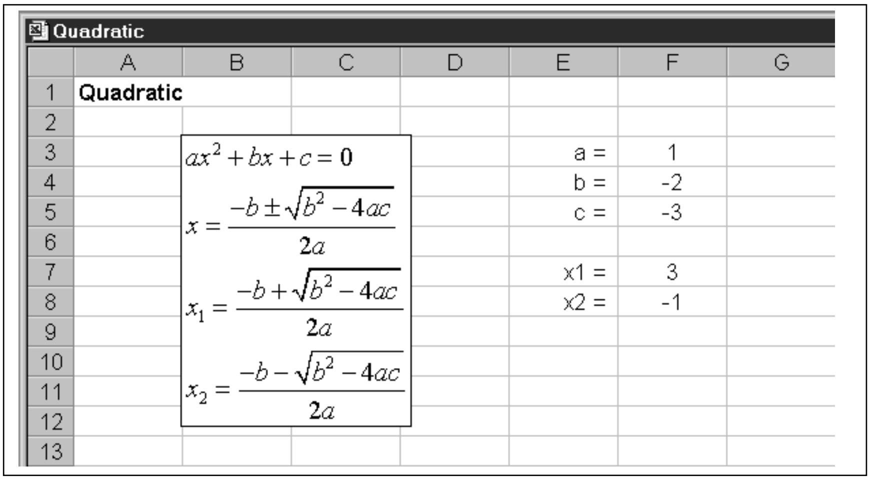
$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \; x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Now set up a spreadsheet with the parameters a, b and c, as shown below. Let their values be placed in cells F3, F4 and F5. (To the left we insert the formulas as a reminder.) Then in cells F7 and F8 place the results, i.e.

F7 $= \left(-b + \sqrt{b^2 - 4ac}\right)/2a$
$= (-\$F\$4 + SQRT(\$F\$4\text{^}2 - 4*\$F\$3*\$F\$5))/(2*\$F\$3)$

F8 $= \left(-b - \sqrt{b^2 - 4ac}\right)/2a$
$= (-\$F\$4 - SQRT(\$F\$4\text{^}2 - 4*\$F\$3*\$F\$5))/(2*\$F\$3)$

Save this spreadsheet. It can now be used to solve any quadratic of the form $ax^2 + bx + c = 0$



The cobweb representation of this nonlinear system is shown in figure 1.6, where we have the curve $2-\frac{1}{2}x^2(t)$ and the 45°-line denoting $x(t+1)=x(t)$.

In order to investigate what is happening in the neighbourhood of the fixed points let us set this problem up on a spreadsheet in just the same manner as our linear example, as shown in figure 1.7. Once again we set this up in general terms, placing the parameters *a* and *c* above the data we are deriving. Also included are the formulas for the two fixed points of the system. These are entered in cells E2 and E3 with the instructions

$$(1+SQRT(1-4*\$C\$2*\$C\$3))/(2*\$C\$3)$$
$$(1-SQRT(1-4*\$C\$2*\$C\$3))/(2*\$C\$3)$$

We next place the initial value in cell B6, which is here equal to 1.25.

In considering what to place in cell B7, consider the algebraic representation and the spreadsheet representation of the problem

$$=2-\tfrac{1}{2}x^2(0)$$
$$=\$C\$2+\$C\$3*B6\char94 2$$

Notice that cell C3 includes the minus sign and that we specify powers in spreadsheets by using the 'caret' symbol. Although the system we are investigating is more complex, there is fundamentally no difference in the way we set it up on the spreadsheet. We can now copy cell B7 to the clipboard and then copy down for as many periods as we wish. To verify we have done all this correctly, copy one of the equilibrium values and place it in cell B6 for the initial value. If all is OK, then *all* values should be 1.23607 (or approximately so depending on the decimal places you have specified for your results). Having performed this test satisfactorily, replace $x(0)$ by 1.25 once again and then experimentation can begin.

1.8.1 A change in the initial value

Let us consider first the lower equilibrium point, $x_1^* = -3.23607$ and an initial value of $x(0) = -3.5$. Given this initial value, the system declines very rapidly,

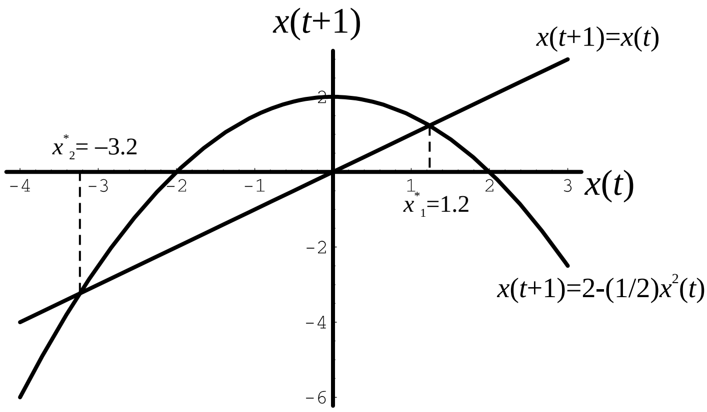


Figure 1.6

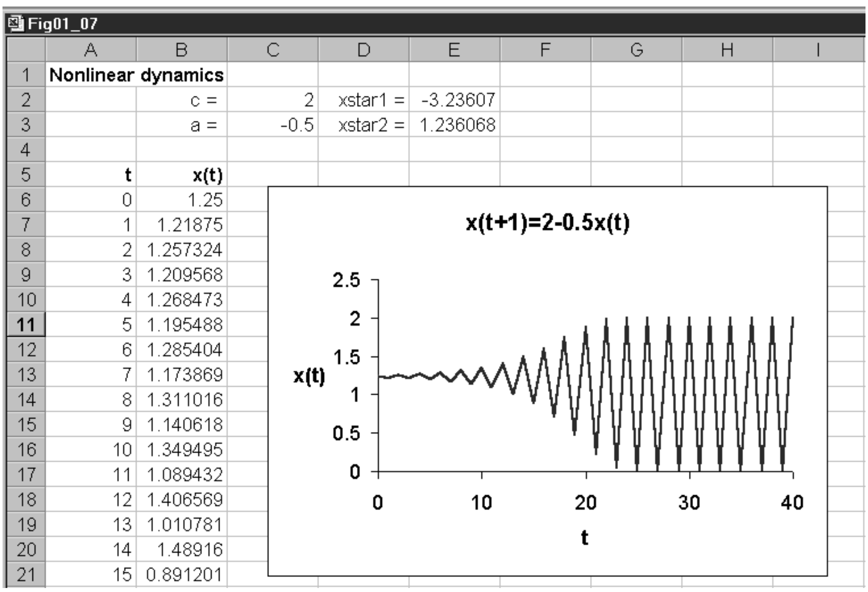


Fig01_07

	A	B	C	D	E	F	G	H	I
1	Nonlinear dynamics								
2		c =	2	xstar1 =	-3.23607				
3		a =	-0.5	xstar2 =	1.236068				
4									
5	t	x(t)							
6	0	1.25							
7	1	1.21875							
8	2	1.257324							
9	3	1.209568							
10	4	1.268473							
11	5	1.195488							
12	6	1.285404							
13	7	1.173869							
14	8	1.311016							
15	9	1.140618							
16	10	1.349495							
17	11	1.089432							
18	12	1.406569							
19	13	1.010781							
20	14	1.48916							
21	15	0.891201							

Figure 1.7

moving further in the negative direction. What about a value slightly larger than -3.2? Consider the value $x(0)=-3.1$. The system certainly moves away from the fixed point, but then begins to oscillate between the values 0 and 2. For the moment we shall not concern ourselves with the oscillatory behaviour, and we shall take up this point later. All we are establishing here is that for initial values a little larger than -3.23607 the system moves away from it, which it certainly does. Hence, the fixed point $x_1^*=-3.23607$ is *locally* unstable.

What about the fixed point $x_2^*=1.23607$? Consider first a value 0.9. Very soon the system settles into an oscillatory behaviour, oscillating once again between 0 and 2. Consider an initial point above $x_2^*=1.23607$, say 1.5. The system once again converges on the oscillation between 0 and 2. What if we

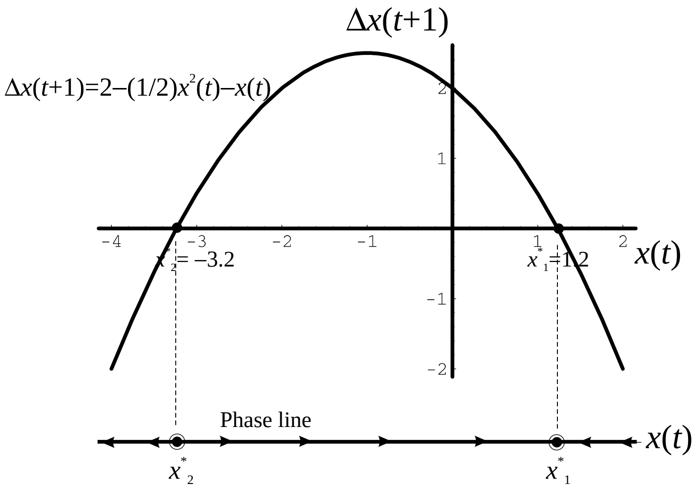


Figure 1.8

choose values even closer to the fixed point? Consider values 1.2 and 1.25, respectively. With initial value 1.2 the system once again settles down to the cycle 0 and 2 by about period 25. With initial point 1.25 the system settles down to the same cycle by about period 30. The fixed point $x_2^* = 1.23607$ is neither an attractor nor a repellor.

In order to see what is taking place let us consider the difference equation version of the model. This is

$$\Delta x(t+1) = 2 - \tfrac{1}{2}x^2(t) - x(t) \quad x(0) = x_0 \tag{1.15}$$

In equilibrium $\Delta x(t+1) = 0$ and so

$$2 - \tfrac{1}{2}x^{*2} - x^* = 0$$
$$\text{or } x^{*2} + 2x^* - 4 = 0$$

with solutions

$$x_1^* = -1 + \sqrt{5} \quad x_2^* = -1 - \sqrt{5}$$

The same equilibrium points have once again been established. The phase diagram representation of the problem is drawn in figure 1.8. The curve represents the equation $2 - \frac{1}{2}x^2(t) - x(t)$. Here we are treating the curve as continuous. ***This is important***. To the left of $x_2^* = -3.23607$ $\Delta x(t+1) < 0$, which indicates that $x(t)$ is falling, so the system is moving even further in the negative direction. Slightly to the right of $x_2^* = -3.23607$ then $\Delta x(t+1) > 0$ and so $x(t)$ is rising, i.e. moving away from the fixed point. From this perspective the fixed point $x_2^* = -3.23607$ is locally unstable and is a repellor.

Now turn to the larger of the fixed points, $x_1^* = 1.23607$. Slightly to the left of this fixed point, in its neighbourhood, $\Delta x(t+1) > 0$ and so $x(t)$ is rising. Slightly to the right of the fixed point, $\Delta x(t+1) < 0$ and so $x(t)$ is falling. In terms of this *continuous* representation it appears that the fixed point $x_1^* = 1.23607$ is locally stable and is an attractor. But this seems in contradiction to our spreadsheet investigation – at least for the higher fixed point! Why is this?

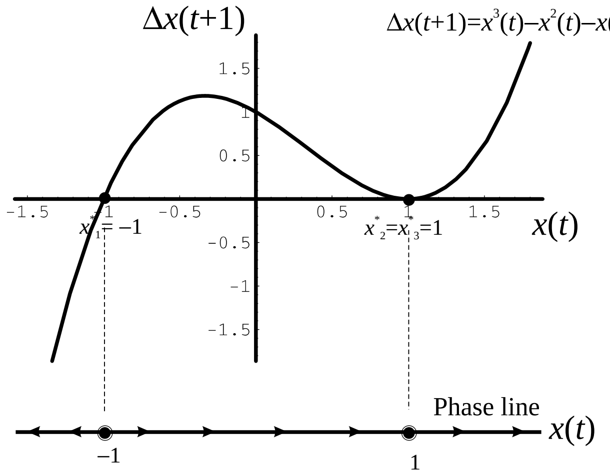


Figure 1.9

What has been illustrated here is that the properties of continuous models are not necessarily the same for their discrete counterpart. In fact, for many discrete nonlinear models oscillatory behaviour arises. We shall see why in later chapters. For the present all we wish to do is point out that if you are interested in continuous models, then the difference representation of the model with its accompanying phase line is sufficient to establish fixed points and their local stability or instability. If, however, the model is in discrete time, then it should be investigated on a spreadsheet to establish whether some of the fixed points exhibit oscillations.

Let us take one further example to illustrate these points. Consider the non-linear recursive model

$$x(t+1) = x^3(t) - x^2(t) + 1 \quad x(0) = x_0 \tag{1.16}$$

The difference form of the model is

$$\Delta x(t+1) = x^3(t) - x^2(t) - x(t) + 1 \quad x(0) = x_0 \tag{1.17}$$

In equilibrium $\Delta x(t+1) = 0$ and so we have

$$x^{*3} - x^{*2} - x^* + 1 = 0$$
$$\text{or } (x^* - 1)^2(1 + x^*) = 0$$

Since the difference equation is to the power three, then there are three solutions to this equation. These are

$$x_1^* = -1, \quad x_2^* = 1, \quad x_3^* = 1$$

The reason why two fixed points are the same is clearly shown in figure 1.9, which plots the equation $x^3(t) - x^2(t) - x(t) + 1$ and shows the phase line below.

Consider first the continuous form of the model as shown in figure 1.9. To the left of $x^* = -1$ $\Delta x(t+1) < 0$ and so $x(t)$ is falling. To the right of $x^* = -1$ $\Delta x(t+1) > 0$ and so $x(t)$ is rising. The fixed point $x^* = -1$ is locally unstable and is a repellor. Now consider the fixed point $x^* = 1$. To the left of this point $\Delta x(t+1) > 0$ and so $x(t)$ is rising. To the right of $x^* = 1$ $\Delta x(t+1) > 0$ and so $x(t)$

is still rising. The unusual nature of the fixed point $x^*=1$ is shown by the phase line with its arrows. The arrows are moving towards the fixed point $x^*=1$ and then away from it to the right. It is as if the system is being 'shunted along'. For this reason, the fixed point $x^*=1$ is referred to as a shunt.

Does the discrete form of the model reveal these properties? In setting up the model on a spreadsheet simply enter the initial value for the variable x, and then in the cell immediately below the initial value, type in the formula – moving the cursor to the cell above when placing in the variable x. Then copy this cell to the clipboard and paste down for as many periods as you wish. Doing this reveals the following. A value to the left of -1, say -1.2, leads the system ever more in the negative direction. A value just above -1, say -0.9, leads the system towards the upper fixed point $x^*=1$. Taking a value just to the left of the upper fixed point, say 0.5, leads the system to the fixed point $x^*=1$. Taking a value just above this fixed point, say 1.1, soon leads the system into ever-higher values. Once again we have verified the same properties for this specific model. In particular, we have illustrated that the lower fixed point is a repellor, and is locally unstable, while the upper fixed point (strictly two) is a shunt. In this particular example, therefore, there is no disparity in the conclusions drawn between the continuous form of the model and the discrete form.

1.9 Continuous models

In section 1.8 we talked about continuous models but used a discrete representation of them. We need to be more precise about continuous models and how to represent them. This is the purpose of this section. In section 1.10 we shall consider a spreadsheet representation of continuous models using Euler's approximation. This will be found especially useful when we consider systems of equations in chapter 4 and later.

If a variable x varies continuously with time, t, then $x(t)$ is a continuous variable. If we know, say from theory, that the change in $x(t)$ over time, denoted $dx(t)/dt$, is

(1.18)
$$\frac{dx(t)}{dt}=f[x(t)]$$

then we have a first-order **differential** equation. If t does not enter explicitly as a separate variable, then the differential equation is said to be an autonomous **differential** equation.[2]

By way of example, suppose

(1.19)
$$\frac{dx(t)}{dt}=4-2x(t)$$

[2] This is the mathematicians' use of the word 'autonomous'. They mean independent of time. When an economist refers to a variable being autonomous they mean being independent of income.